Mathematic 104, Fall 2010: Assignment #2 (v2)

Due: **Wednesday, October 13th**

Instructions: Please ensure that your answers are legible. Also make sure that all steps are shown – even for problems consisting of a numerical answer. Bonus problems cover advanced material and, while good practice, are *not* required and will *not* be graded.

Problem #1. Excercise 1.3 of Lecture 1 of Trefethen-Bau.

Problem #2. Consider the following three vectors in $\mathbb{C}^3$:

$$v_1 = \begin{bmatrix} 3I \\ 0 \\ 4 \end{bmatrix}, v_2 = \begin{bmatrix} 4 \\ 0 \\ 3I \end{bmatrix}, v_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

a) Show that $\{v_1, v_2, v_3\}$ is an orthogonal set. Is this set orthonormal?
b) Let $X \in \mathbb{C}^{3\times 3}$ be a matrix so that $Xv_1 = v_2 + v_3$, $Xv_2 = -v_2$ and $Xv_3 = v_1 + v_2 + v_3$. Determine X. (Hint: Look for a natural orthonormal basis).

Problem #3. Let v_1, v_2 and v_3 be vectors in $\mathbb{C}^3$. Determine a value $\lambda_0 \in \mathbb{C}$ so that when $\lambda = \lambda_0$ the vectors $w_1 = v_1 + v_2, w_2 = v_1 - v_3$ and $w_3 = \lambda v_1 + v_2 + v_3$ are never a basis of $\mathbb{C}^3$. When $\lambda \neq \lambda_0$ what condition on the $\{v_i\}$ is neccesary and sufficient so that the $\{w_i\}$ form a basis?

Problem #4. Let u and v be two vectors in $\mathbb{R}^3$ so that $||u||_2 = 5$ and $||v||_2 = 13$.

a) What are the largest and smallest values of $||2u + v||_2$?
b) What are the largest and smallest values of $\langle u, v\rangle = u^* v$?

Problem #5. Let Q be the following 4×4 matrix:

$$Q = \begin{bmatrix} \frac{3}{5} & 0 & 0 & q_1 \\ 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & q_2 \\ 0 & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & q_3 \\ \frac{4}{5} & 0 & 0 & q_4 \end{bmatrix} \text{ and denote by } q = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} \text{ the fourth column of } Q.$$

a) Assume $Q \in \mathbb{R}^{4\times 4}$ determine *all* $q \in \mathbb{R}^4$ so that Q is orthogonal – that is $Q^{-1} = Q^\top$.
b) Assume instead that $Q \in \mathbb{C}^{4\times 4}$ determine *all* $q \in \mathbb{C}^4$ so that Q is unitary – that is $Q^{-1} = Q^*$.

Bonus Problem. Verify some of the properties of the adjoint stated in class. Recall, we defined the adjoint by letting

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \in \mathbb{C}^{m\times n} \text{ and } B = \begin{bmatrix} b_{11} & \cdots & b_{1m} \\ \vdots & \ddots & \vdots \\ b_{n1} & \cdots & b_{nm} \end{bmatrix} \in \mathbb{C}^{n\times m}$$

and saying B is the adjoint of A when and only when $b_{ij} = \bar{a}_{ji}$ and then writing $A^* = B$.

a) For $A \in \mathbb{C}^{m\times n}$ let $a_i \in \mathbb{C}^m$ be the columns of A, i.e. $A = \begin{bmatrix} a_1 | & \cdots & | a_n \end{bmatrix}$ verify $A^* = \begin{bmatrix} a_1^* \\ \vdots \\ a_n^* \end{bmatrix}$.
b) For $B \in \mathbb{C}^{m\times n}$ let $b_i \in \mathbb{C}^{1\times n}$ be the rows of B that is $B = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$ verify that $B^* = \begin{bmatrix} b_1^* | & \cdots & | b_m^* \end{bmatrix}$.
c) Check that $(A + B)^* = A^* + B^*$, $(\lambda A)^* = \bar{\lambda} A^*$ and $(A^*)^* = A$ for $A, B \in \mathbb{C}^{m\times n}$ and $\lambda \in \mathbb{C}$. (Hint: Consder the rules for matrix additon and scalar multiplication as they apply to each entry).
d) Show that $(AB)^* = B^* A^*$ for $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{n\times k}$. (Hint: Express the product in terms of columns and rows).
e) Show that $\langle Av, w\rangle = \langle v, A^* w\rangle$ for $v \in \mathbb{C}^n$, $w \in \mathbb{C}^m$ and $A \in \mathbb{C}^{m\times n}$.