Problem #1

first, assume $I+uv^*$ is invertible. We suppose the inverse is of the form $I+\alpha uv^*$ and try to find if we succeed then since inverse is unique we will be done.

$$(1)\quad I=(I+uv^*)(I+\alpha uv^*)=I+(\alpha+1)uv^*+\alpha uv^*uv^*$$

$$uv^*=\begin{bmatrix} u_1v_1^* & \cdots & u_1v_m^* \\ \vdots & & \vdots \\ u_mv_1^* & \cdots & u_mv_m^* \end{bmatrix} \qquad uv^*.uv^*=\begin{bmatrix} u_iv_j^*(u_1v_1^*+u_2v_2^* \cdots + u_mv_m^*) \end{bmatrix}$$

so $uv^*.uv^*=(u_1v_1^*+ \cdots + u_mv_m^*).uv^*$

$$(1)\Rightarrow \quad (\alpha+1)uv^*+\alpha(u_1v_1^*+\cdots+u_mv_m^*)uv^*=0$$

Since uv^* is not trivial matrix so $\alpha=\dfrac{-1}{1+u_1v_1^*+\cdots+u_mv_m^*}$

if $I+uv^*$ is singular

then $\exists\, x\neq\vec{0}$ such that $(I+uv^*)x=0$

$uv^*x=-x \Rightarrow \langle v,x\rangle u=-x$ so u has to be in direction x

let say $\vec{u}=k\vec{x}$ then $\langle v,x\rangle kx=-x \Rightarrow k=\dfrac{-1}{\langle v,x\rangle}=\dfrac{-1}{\frac{1}{k}\langle v,u\rangle} \Rightarrow \langle v,u\rangle=-1$

and any $x=ku \in \ker(A)=N(A)$ so $N(A)=\langle u\rangle$

Problem #2

if A^*A is non singular then Obviously A is nonsingular, otherwise

$\exists\, x \neq 0$, $Ax = 0 \implies A^*Ax = 0 \implies \times$ since A^*A is nonsingular.

now assume A is nonsingular if A^*A happens to be singular:

$$\exists\, x \neq 0 \qquad A^*Ax = 0 \implies 0 = \langle A^*Ax, x\rangle \overset{\text{bonus problem}}{\underset{\text{part (e) Problem set 2}}{=\!=}} \langle Ax, Ax\rangle = \|Ax\|^2 \geq 0$$

$$\text{So } \|Ax\|^2 = 0 \implies Ax = 0 \overset{A \text{ is non-sing}}{\implies} x = 0 \quad \text{So } A^*A \text{ is nonsingular}$$

Problem # 3

(a) first we normalize two vectors

$$\frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\1\end{bmatrix}, \quad \frac{1}{\sqrt{2}}\begin{bmatrix}0\\-1\\1\end{bmatrix}$$

$$u_1 \qquad\qquad u_2$$

by Gram-Schmidt we find orthonormal basis for $E = \mathrm{span}(u_1, u_2)$

$$e_1' = u_1, \quad e_2' = u_2 - \langle u_2, u_1\rangle u_1$$

$$e_1' = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\1\end{bmatrix}, \quad e_2' = \frac{1}{\sqrt{2}}\begin{bmatrix}0\\-1\\1\end{bmatrix} - \left\langle \frac{1}{\sqrt{2}}\begin{bmatrix}0\\-1\\1\end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\1\end{bmatrix}\right\rangle \frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\1\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}-\frac{1}{2}\\-1\\\frac{1}{2}\end{bmatrix}$$

normilizing e_1', e_2'

$$e_1 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\1\end{bmatrix} \qquad e_2 = \frac{\sqrt{2}}{\sqrt{3}}\begin{bmatrix}-\frac{1}{2}\\-1\\\frac{1}{2}\end{bmatrix}$$

So by (6.6)

$$P = \hat{Q}\hat{Q}^* = \begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{6}}\\ 0 & \frac{-\sqrt{2}}{\sqrt{3}}\\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}}\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}}\\ \frac{-1}{\sqrt{6}} & \frac{-\sqrt{2}}{\sqrt{3}} & \frac{1}{\sqrt{6}}\end{bmatrix}$$

$$= \begin{bmatrix}\frac{2}{3} & \frac{1}{3} & \frac{1}{3}\\ \frac{1}{3} & \frac{2}{3} & -\frac{1}{3}\\ \frac{1}{3} & -\frac{1}{3} & \frac{2}{3}\end{bmatrix}$$

(b) $$I - P = \frac{1}{3}\begin{bmatrix}1 & -1 & -1\\ -1 & 1 & 1\\ -1 & 1 & 1\end{bmatrix} = P_q = qq^T = \begin{bmatrix}q_1^2 & q_1q_2 & q_1q_3\\ q_1q_2 & q_2^2 & q_2q_3\\ q_1q_3 & q_3q_2 & q_3^2\end{bmatrix}$$

So $q = \frac{1}{\sqrt{3}}(1, -1, -1)$ works.

Problem # 3

(a) by Gram-Schmidt we find a orthonormal basis of

$$\text{Span} \left\langle \underbrace{\frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\1\end{bmatrix}}_{\frac{u_1}{\|u_1\|}}, \underbrace{\frac{1}{\sqrt{2}}\begin{bmatrix}0\\-1\\1\end{bmatrix}}_{\frac{u_2}{\|u_2\|}} \right\rangle .$$

$$v_1 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\1\end{bmatrix}$$

$$v_2 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\1\end{bmatrix} - \langle$$

Problem #5

(a) from (6.8) we know that $P_v = v.v^*$ so

$$P_v = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \end{bmatrix} = \begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \sin\theta\cos\theta & \sin^2\theta \end{bmatrix}$$

(b) We can compute projection on standard basis and then change the basis by Rotation $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = U$

$$P_{(1,0)} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1, & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = X$$

$$U \qquad X \qquad U^T$$

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} =$$

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \cos^2\theta & \sin\theta\cos\theta \\ \sin\theta\cos\theta & \sin^2\theta \end{bmatrix}$$

$$= P_v$$

Problem # 4

$$\mathbb{C}^k \xrightarrow{B} \mathbb{C}^n \xrightarrow{A} \mathbb{C}^m$$

first assume $R(AB) = R(A)$ if $N(A) + R(B) \neq \mathbb{C}^n$ then there exists

$x \in \mathbb{C}^n$ such that $x \notin N(A) + R(B)$

Since $R(AB) = R(A)$ so $\exists$ $y \in \mathbb{C}^k$ such that

$$ABy = Ax \implies A(-By + x) = 0 \implies x - By \in N(A)$$

so $x \in N(A) + By \subset N(A) + R(B)$ $\times$

so $R(B) + N(A) = \mathbb{C}^n$

now assume $R(B) + N(A) = \mathbb{C}^n$. Obviously $R(AB) \subseteq R(A)$ assume

$\exists x$ such that $x \in R(A)$, $x \notin R(AB)$ so take $y \in \mathbb{C}^n$ s.t

$Ay = x$, but since $y \in \mathbb{C}^n = N(A) + R(B)$ so $\exists r, s$, $r \in N(A)$, $s \in R(B)$

such that $r + s = y$ but $A(r+s) = \overset{0}{\cancel{Ar}} + As = Ay = x \implies As = Ay = x$

$s \in R(B) \implies \exists$ $b \in B$ s.t $Bb = s \implies ABb = As = x \implies x \in R(AB)$ $\times$

so $R(A) = R(AB)$

Bonus Problem:

note that if we choose a basis for a subvector space $W \subset V$ then we can extend that basis to a basis of V

So it is trivial that $E \cap F$ is subspace of E and F so choose a basis $\{e_1, \dots, e_r\}$ for $E \cap F$ so $\dim(E \cap F) = r$

now extend $\{e_1, \dots, e_r\}$ to a basis of E, let say $\{e_1, \dots, e_r, v_1, \dots, v_k\}$

and " " " " " " F, " " $\{e_1, \dots, e_r, w_1, \dots, w_\ell\}$

So $\dim(E) = r+k$ and $\dim(F) = r+\ell$. now we show that

$\{e_1, \dots, e_r, v_1, \dots, v_k, w_1, \dots, w_\ell\}$ is a basis for $E+F$. if we prove our claim then $\dim(E+F) = r+k+\ell = \dim(E) + \dim(F) - \dim(E \cap F)$

So we just need to show our claim. If $\exists \alpha_1, \dots, \alpha_r, \beta_1, \dots, \beta_k, \gamma_1, \dots, \gamma_\ell \in \mathbb{C}$

s.t
$$\sum_{i=1}^{r} \alpha_i e_i + \sum_{i=1}^{k} \beta_i v_i + \sum_{i=1}^{\ell} \gamma_i w_i = 0 \Rightarrow \sum \alpha_i e_i + \sum \beta_i v_i = -\sum \gamma_i w_i$$

the left hand side is in E and Righ hand side is in W so both should be in $E \cap F$ so $\exists \alpha_i'$ s.t:

Bonus Problem

~~assume $v_1, \dots, v_k$ are basis of E and $w_1, \dots, w_\ell$ are basis of F.~~

$-\sum_{i=1}^{l} \gamma_i w_i = \sum_{i=1}^{r} \alpha_i' e_i \implies \sum_{i=1}^{r} \alpha_i' e_i + \sum_{i=1}^{l} \gamma_i w_i = 0 \implies \alpha_i' = 0 , \gamma_i = 0$ for

all i since $\{e_1, -, e_r, w_1, -, w_l\}$ is a basis for W

and $\sum \alpha_i e_i + \sum \beta_i v_i = \sum \alpha_i' e_i \implies \sum_{i=1}^{r} (\alpha_i - \alpha_i') e_i + \sum_{i=1}^{k} \beta_i v_i = 0 \implies$

$\forall i,j \quad \alpha_i = \alpha_i' = 0 , \beta_j = 0$ since $\{e_1, -, e_r, v_1, -, v_k\}$ is a basis for

E. So all $\alpha_i, \beta_i, \gamma_i$ are zero so $\{e_1, -, e_r, v_1, -, v_k, w_1, -, w_l\}$ are

independent but they obviously generate $E+F$ so it is a basis.