Mathematic 104, Fall 2010: Assignment #5

Due: **Wednesday, November 10th**

Instructions: Please ensure that your answers are legible. Also make sure that all steps are shown – even for problems consisting of a numerical answer. Bonus problems cover advanced material and, while good practice, are *not* required and will *not* be graded.

Problem #1. Let $A \in \mathbb{R}^{2\times 2}$ be the following matrix

$$A = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$$

Denote by $S_p = \{u \in \mathbb{R}^2 : ||u||_p = 1\}$ the set of vectors of length 1 in the p-norm and let $AS_p = \{Au \in \mathbb{R}^2 : u \in S_p\}$ be the image under A of S_p. Here $1 \leq p \leq \infty$.

a) Determine $||A||_1$ and find vectors $u \in S_1$ and $v = Au \in AS_1$ so that $||v||_1 = ||Au||_1 = ||A||_1$. Sketch S_1 and AS_1 and indicate the vectors u and v on the sketch.
b) Determine $||A||_\infty$ and find vectors $u \in S_\infty$ and $v = Au \in AS_\infty$ so that $||v||_\infty = ||Au||_\infty = ||A||_\infty$. Sketch S_∞ and AS_∞ and indicate the vectors u and v on the sketch.
c) Determine $||A||_2$ and find vectors $u \in S_2$ and $v = Au \in AS_2$ so that $||v||_2 = ||Au||_2 = ||A||_2$. Sketch S_2 and AS_2 and indicate the vectors u and v on the sketch. To do this it is useful to use that any vector $u \in S_2$ may be written as
$$u = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}.$$

Problem #2. Suppose that $P \in \mathbb{C}^{m\times m}$ and $P^2 = P$ (so P is an oblique projector). Show that as long as $P \neq 0$, $||P||_2 \geq 1$ and $||P||_2 = 1$ when and only when P is an orthogonal projector.

Problem #3. Compute $||P||_F$ when P is an orthogonal projector. (Hint: Use the proof of Theorem 6.1 of Trefethen-Bau).

Problem #4. Excercise 3.5 of Lecture 3 of Trefethen-Bau.

Bonus Problem. Consider X the set of points $(x, y) \in \mathbb{R}^2$ that statisfy

$$Ax^2 + Bxy + Cy^2 + Dy + Ex + F = 0$$

for $A, B, C, D, E, F \in \mathbb{R}$ and A, B, C not all zero. This set is called a *conic section* and X may (among other things) be an ellipse, a hyperbola or a parabola. It turns out that if $B^2 - 4AC < 0$ then X either consists of zero or one point or is an ellipse. Assume the points $(x_1, y_1), \dots, (x_n, y_n) \in \mathbb{R}^2$ are known to lie approximately on an ellipse. Set up a least squares problem to find X the ellipse that best fits these points in the sense of least squares.