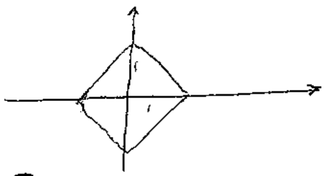
problem #1 : $A = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$

a) We know that $\|A\|_1 = \sup \frac{\|Au\|_1}{\|u\|_1} = \sup_{\|u\|_1=1} \|Au\|_1$

$A\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} u_1+2u_2 \\ -u_2 \end{bmatrix} \Rightarrow \|Au\|_1 = |u_1+2u_2| + |-u_2| \le |u_1| + 3|u_2| = 1+2|u_2| \le 3$ since

we know that $|u_1|+|u_2|=1$ so $|u_2|\le 1$ so $\|Au\|_1 \le 3$ and for $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ it takes sup value

so $u = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $Au = v = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$



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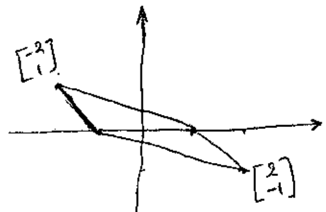

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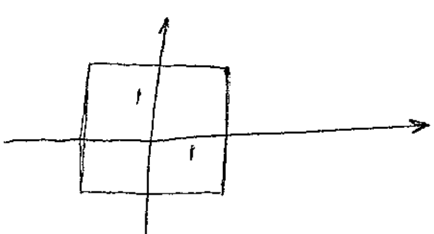
b) by definition we know that $\|A\|_\infty$ equals 1-norm of rows of A, but A has two rows $[1,2]$ and $[0,-1]$ and their norms are 3, 1 respectively.

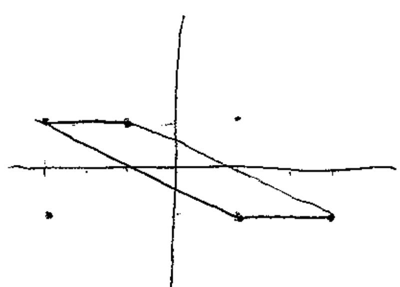
so $\|A\|_\infty = 3$ and assume $u = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ then $Au = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$ and

$\|Au\|_\infty = \max\{|3|, |-1|\} = 3$ (note that $\|u\|_\infty = 1$) so we found u such

that $\|Au\|_\infty = \|A\|_\infty$.



S_∞



AS_∞

c) We try to find $\|A\|_2$ by verifying Au where $u = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$ so:

$$Au = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} = \begin{bmatrix} \cos\theta + 2\sin\theta \\ -\sin\theta \end{bmatrix} \Longrightarrow \|Au\|_2^2 = \cos^2\theta + 4\sin^2\theta + 4\sin\theta\cos\theta + \sin^2\theta$$

$$= 1 + 4\sin\theta(\sin\theta + \cos\theta)$$

So $\|A\|_2 = \sup\limits_{\|u\|_2 = 1} \|Au\|_2$

So $$\frac{\partial(\|Au\|_2^2)}{\partial\theta} = 4\cos\theta(\sin\theta + \cos\theta) + 4\sin\theta(\cos\theta - \sin\theta) = 8\sin\theta\cos\theta + 4\cos 2\theta$$

$$= 4\sin 2\theta + 4\cos 2\theta$$

So for max $\dfrac{\partial(\|Au\|_2^2)}{\partial\theta} = 0 \Longrightarrow \sin 2\theta = -\cos 2\theta \Longrightarrow \tan 2\theta = -1$ so

$$2\theta = k\pi + \frac{3\pi}{4} \quad \text{so} \quad \theta = k\frac{\pi}{2} + \frac{3\pi}{8}$$ takes $k=0$ for example

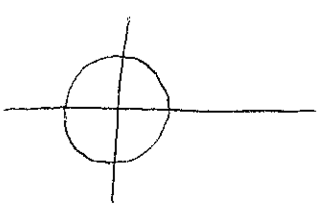
We get $\theta = \dfrac{3\pi}{8}$ and $\|Au\|_2^2 = 4\sin^2\dfrac{3\pi}{8} + 4\sin\dfrac{3\pi}{8}\cos\dfrac{3\pi}{8} = 2\left(1 - \cos\dfrac{3\pi}{4}\right) + 2\sin\dfrac{3\pi}{4}$

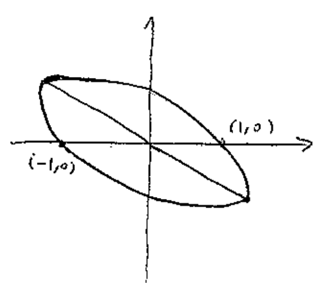
$$= 2\left(1 + \frac{\sqrt{2}}{2}\right) + 2\frac{\sqrt{2}}{2} = 4\sqrt{2} + 2$$

(you can easily check with second derivative criteria that $\frac{3\pi}{8}$ gives max not min)

So the max $\|Au\|_2$ is $\sqrt{4\sqrt{2} + 2}$ so $\|A\|_2 = \sqrt{4\sqrt{2} + 2}$

and u that $\|Au\|_2 = \|A\|_2$ is $\begin{bmatrix} \cos\frac{3\pi}{8} \\ \sin\frac{3\pi}{8} \end{bmatrix}$





problem #2 : if P is projector then since $\|P\|_2 = \sup_{\|u\|=1} \|Pu\|_2$ so

$\|P\|_2 \geq \|Pu\|_2$ $(\|u\|_2 = 1)$ So choose a unit vector in range P then $u \in \text{range}(P)$

so $Pu = u$ so $\|Pu\|_2 = \|u\|_2 = 1$ So $\|P\|_2 \geq 1$ *

now assume that P is orthogonal projector then We know that $\text{range}(P) \perp \text{Ker}(P)$

So for any vector u We have:

$$u = \underbrace{(u - Pu)}_{\in \text{Ker } P} + \underbrace{Pu}_{\in \text{range}(P)} \Longrightarrow \|u\|_2^2 = \|u - Pu\|_2^2 + \|Pu\|_2^2 \geq \|Pu\|_2^2$$

So $\dfrac{\|Pu\|_2}{\|u\|_2} \leq 1 \Longrightarrow \|P\|_2 \leq 1$ but by * We know $\|P\|_2 \geq 1$ So $\|P\|_2 = 1$

now assume $\|P\|_2 = 1$. in order to show P is orthogonal projector again We have to show $\text{range}(P) \perp \text{Ker}(P)$ So assume Contrary Then it means that there exists u such that $u \perp \text{Ker } P$ but $u \notin \text{range}(P)$ so

$$P(Pu - u) = 0 \Longrightarrow Pu - u \in \text{Ker } P \Longrightarrow u \perp Pu - u \Longrightarrow Pu = \underbrace{(Pu - u)}_{\in \text{Ker } P} + \underbrace{u}_{\in \text{Ker } P^{\perp}}$$

So $\|Pu\|_2^2 = \|Pu - u\|_2^2 + \|u\|_2^2 \Longrightarrow \|Pu\|_2 > \|u\|_2$ So $\|P\|_2 > 1$ but We know that $\|P\|_2 = 1$. Contradiction so range $P \perp \text{Ker } P$ So

P is orthogonal projector.

problem #4, assume $u=[u_1,-,u_m]$ and $v=[v_1,-,v_n]$

$$\|u\|_F^2 = \sum_{i=1}^{m} |u_i|^2 \qquad \|v\|_F^2 = \sum_{i=1}^{n} |v_i|^2$$

$$uv^* = \begin{bmatrix} u_1\bar{v}_1 & u_1\bar{v}_2 & \cdots & \\ u_2\bar{v}_1 & & \ddots & \\ \vdots & & & \\ u_m\bar{v}_1 & & \cdots & u_m\bar{v}_n \end{bmatrix} \quad \text{so} \quad \|uv^*\|_F^2 = \sum_{i,j} |u_i\bar{v}_j|^2$$

but $|u_i\bar{v}_j| = |u_i||\bar{v}_j| = |u_i||v_j|$ so

$$\|u\|_F^2\,\|v\|_F^2 = \left(\sum_{i=1}^{m} |u_i|^2\right)\left(\sum_{j=1}^{n} |v_j|^2\right) = \sum_{i,j} |u_i||v_j|^2 = \sum_{i,j} |u_i\bar{v}_j|^2 = \|uv^*\|_F$$

so $\|u\|_F\,\|v\|_F = \|uv^*\|_F$