Mathematic 104, Fall 2010: Assignment #6

Due: **Wednesday, November 17th**

Instructions: Please ensure that your answers are legible. Also make sure that all steps are shown – even for problems consisting of a numerical answer. Bonus problems cover advanced material and, while good practice, are *not* required and will *not* be graded.

Problem #1. Consider the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$$

a) By hand compute the SVD of A. To do this it is useful to recall that all vectors in x with $||x||_2 = 1$ are of the form

$$x = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}.$$

b) Using the SVD determine the rank one matrix B that best approximates A in the Frobenius norm.

c) Compare how well B approximates A in the Frobenius norm with how well the rank one matrices

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \text{ and } A_2 = \begin{bmatrix} 0 & 2 \\ 0 & 2 \end{bmatrix}$$

approximate A in the Frobenius norm.

Problem #2. Excercise 4.4 of Lecture 4 of Trefethen-Bau.

Problem #3. Let $A_1, A_2 \in \mathbb{C}^{m\times m}$ suppose that the left singular vectors of A_1 are $\{u_1^1, \dots, u_m^1\}$ and the right singular vectors are $\{v_1^1, \dots, v_m^1\}$ while the left singular vectors of A_2 are $\{u_1^2, \dots, u_m^2\}$ and the right singular vectors are $\{v_1^2, \dots, v_m^2\}$. Show that if for $1 \le i \le m$

$$u_i^1 = v_i^1 = u_i^2 = v_i^2$$

then $A_1A_2 = A_2A_1$. That is if two square matrices have the same left and right singular vectors then they commute. (Hint: Think about the problem for diagonal matrices first, then use the SVD of A_1 and A_2).

Problem #4. Suppose that A has the following (full) SVD

$$A = \begin{bmatrix} 0 & \sqrt{2}/2 & \sqrt{2}/2 \\ 1 & 0 & 0 \\ 0 & -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 4/5 & -3/5 \\ 3/5 & 4/5 \end{bmatrix}$$

a) Using only the SVD give an orthonormal basis of $R(A)$.

b) Compute a (full) QR factorization of A.

c) Using the QR factorization of A give an orthonormal basis of $R(A)$. Does this agree with your answer in a)?

Problem #5. Let $A \in \mathbb{C}^{m\times m}$ have singular values $\sigma_1 \ge \sigma_2 \ge \dots \ge \sigma_m \ge 0$.

a) Show that if $\lambda = \mu\sigma_1$ is an eigenvalue of A for $|\mu| = 1$ then $||A^2||_2 = ||A||_2^2$. Recall λ is an eigenvalue of A when the matrix $A - \lambda I$ is singular.

b) Assume in addition that $\sigma_1 > \sigma_2$ show that if $||A^2||_2 = ||A||_2^2$ then $\mu\sigma_1$ is an eigenvalue of A for some μ with $|\mu| = 1$. (Hint: Think about the proof of the uniqueness of singular vectors for the SVD)

Bonus Problem. Determine if the condition in Problem # 5 part b) that $\sigma_1 > \sigma_2$ is necessary. That is prove the result without this assumption or give a counter-example.