**1 (40 pts, 5 for each)** Evaluate the following limits

a) $\lim_{x\to -3} \frac{x^2+5x+6}{x^2+4x+3}$

$$\frac{x^2+5x+6}{x^2+4x+3} = \frac{(x+2)(x+3)}{(x+1)(x+3)} = \frac{x+2}{x+1} \quad \forall x \neq -3$$

$$\Rightarrow \lim_{x\to -3} \frac{x^2+5x+6}{x^2+4x+3} = \lim_{x\to -3} \frac{x+2}{x+1} = \frac{-3+2}{-3+1} = \frac{-1}{-2} = \frac{1}{2}$$

b) $\lim_{x\to\infty} \frac{x^2+5x+6}{x^2+4x+3}$

i) $$\lim_{x\to\infty} \frac{x^2+5x+6}{x^2+4x+3} = \lim_{x\to\infty} \frac{x+2}{x+1} = \lim_{x\to\infty} \frac{1+2/x}{1+1/x} = \frac{1}{1} = 1$$

ii) $$\lim_{x\to\infty} \quad " \quad = \lim_{x\to\infty} \frac{1+5/x+6/x^2}{1+4/x+3/x^2} = \frac{1+0+0}{1+0+0} = 1$$

c) $\lim_{x\to 0} \frac{x^2+5x+6}{x^2+4x+3}$

$x^2+4x+3$ continuous and equals 3 at $x=0$

$$\therefore \lim_{x\to 0} \frac{x^2+5x+6}{x^2+4x+3} = \frac{0^2+5\cdot 0+6}{0^2+4\cdot 0+3} = \frac{6}{3} = 2$$

d) $\lim_{x\to\infty}(\sqrt{x^2+4} - x)$

$\infty - \infty$ type

$$\sqrt{x^2+4} - x = \frac{(\sqrt{x^2+4}-x)(\sqrt{x^2+4}+x)}{\sqrt{x^2+4}+x} = \frac{4}{\sqrt{x^2+4}+x} = \frac{4/x}{\sqrt{1+4/x^2}+1}$$

$$\therefore \lim_{x\to\infty}(\sqrt{x^2+4} - x) = \lim_{x\to\infty} \frac{4/x}{\sqrt{1+4/x^2}+1} = \frac{0}{1+1} = 0$$

e) $\lim_{x\to 0} e^{x\cos(e^{-1/x})}$

$\lim_{x\to 0} e^{x\cos(e^{-1/x})} = e^{\lim_{x\to 0} x\cos(e^{-1/x})}$ since $y=e^x$ is continuous

$-1 \le \cos u \le 1$, $\lim_{x\to 0} x = 0 \Rightarrow \lim_{x\to 0} x\cos(e^{-1/x}) = 0$ by squeeze theorem

$\therefore \lim_{x\to 0} e^{x\cos(e^{-1/x})} = e^{\lim_{x\to 0} x\cos(e^{-1/x})} = e^0 = 1$

f) $\lim_{x\to\infty} \frac{1}{x\sin(2/x)}$

i) Let $u = \frac{2}{x}$, then $x = \frac{2}{u}$ $\lim_{x\to\infty} u = 0 \Rightarrow \lim_{x\to\infty} \frac{1}{x\sin\frac{2}{x}} = \lim_{u\to 0^+} \frac{u}{2\sin u} = \frac{1}{2}\lim_{u\to 0^+}\frac{u}{\sin u} = \frac{1}{2}$

ii) $\frac{1}{x\sin(2/x)} = \frac{1/x}{\sin(2/x)}$ type $0/0$

L'Hospital $\lim_{x\to\infty} \frac{1}{x\sin(2/x)} = \lim_{x\to\infty} \frac{1/x}{\sin(2/x)} = \lim_{x\to\infty} \frac{-\frac{1}{x^2}}{\cos(2/x)\cdot\left(-\frac{2}{x^2}\right)} = \lim_{x\to\infty} \frac{1}{2\cos(2/x)} = \frac{1}{2}$

g) $\lim_{x\to\infty}(1+\frac{1}{2x})^x$

type $1^\infty$, $\ln y = \ln\left(1+\frac{1}{2x}\right)^x = x\ln\left(1+\frac{1}{2x}\right) = \frac{\ln(1+\frac{1}{2x})}{1/x}$ type $0/0$

$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{\ln(1+\frac{1}{2x})}{1/x} = \lim_{x\to\infty} \frac{\frac{1}{1+\frac{1}{2x}}\left(-\frac{1}{2x^2}\right)}{-\frac{1}{x^2}} = \lim_{x\to\infty} \frac{1}{2+\frac{1}{x}} = \frac{1}{2}$

$\therefore \lim_{x\to\infty} y = \lim_{x\to\infty} e^{\ln y} = e^{\lim_{x\to\infty}\ln y} = e^{\frac{1}{2}}$

h) $\lim_{x\to\infty} x^{1/x}$

type $\infty^0$, $y = x^{\frac{1}{x}}$, $\ln y = \ln x^{\frac{1}{x}} = \frac{1}{x}\ln x = \frac{\ln x}{x}$ $\frac{\infty}{\infty}$

$\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{1/x}{1} = 0$

$\Rightarrow \lim_{x\to\infty} y = \lim_{x\to\infty} e^{\ln y} = e^{\lim_{x\to\infty}\ln y} = e^0 = 1$

**2 (30 pts, 6 for each.)** Find the derivatives $y' = f'(x)$ of the following functions $y = f(x)$.

a) $f(x) = \ln(\sin(x^2+1))$

Let $y = \ln u \quad u = \sin w \quad w = x^2+1$

$$y' = \frac{dy}{dx} = \frac{dy}{du}\,\frac{du}{dw}\,\frac{dw}{dx} = \frac{1}{u}(\cos w)(2x) = \frac{\cos(x^2+1)}{\sin(x^2+1)}\,2x = 2x\cot(x^2+1)$$

b) $f(x) = (x^2+1)\tan^{-1}(x)$

we know $\frac{d}{dx}\tan^{-1}(x) = \frac{1}{1+x^2}$

$\therefore$ product rule $\Rightarrow$ $f'(x) = (x^2+1)\cdot\frac{1}{x^2+1} + (2x)(\tan^{-1}(x)) = 1 + 2x\tan^{-1}(x)$

c) $f(x) = (\cos x)^{\sin x}$ $(x \in (0, \pi/2))$

$$y = (\cos x)^{\sin x} \Rightarrow \ln y = \sin x \ln(\cos x)$$

$$\Rightarrow \frac{y'}{y} = \cos x \ln(\cos x) + \sin x \frac{-\sin x}{\cos x}$$

$$= \cos x \ln(\cos x) - \frac{\sin^2 x}{\cos x}$$

$$\therefore y' = y\cdot\frac{y'}{y} = (\cos x)^{\sin x + 1}\ln(\cos x) - (\cos)^{\sin x - 1}\sin^2 x$$

d) $\int_{\ln x}^{x} \sin(e^t)dt$

$$\int_{\ln x}^{x} \sin(e^t)dt = \int_0^x \sin(e^t)dt - \int_0^{\ln x} \sin(e^t)dt$$

Let $u = \ln x \Rightarrow$

$$\frac{d}{dx}\int_{\ln x}^{x} \sin(e^t)dt = \frac{d}{dx}\int_0^x \sin(e^t)dt - \frac{d}{dx}\int_0^u \sin(e^t)dt$$

$$\overset{FTC I}{=\!=} \sin(e^x) - \frac{d}{du}\left(\int_0^u \sin(e^t)dt\right)\frac{du}{dx}$$

$$= \sin(e^x) - \sin(e^u)\frac{1}{x} = \sin(e^x) - \frac{\sin x}{x}$$

e) $xy + \ln y = 2x^2 + y^2 + 3$

Implicit differentiation

$$xy' + y + \frac{1}{y}y' = 4x + 2yy' + 0$$

$$\Rightarrow \left(x + \frac{1}{y} - 2y\right)y' = 4x - y$$

$$\Rightarrow \quad y' = \frac{4x-y}{x+\frac{1}{y}-2y} = \frac{4xy-y^2}{xy+1-2y^2}$$

3 **(40 pts, 10 for each.)** Evaluate the following integrals.

a) $\int (\sec\theta)^2 \tan\theta d\theta$

Let $u = \tan\theta$, $du = \sec^2\theta \, d\theta$

$\therefore \int \sec^2\theta \tan\theta \, d\theta = \int u \, du = \frac{u^2}{2} + C = \frac{\tan^2\theta}{2} + C$

b) $\int_{-1}^{1} \frac{x^3}{1+x^4} dx$

i) Let $u = 1+x^4$ then $du = 4x^3 dx$ $\quad x^3 dx = \frac{du}{4}$

$u(-1) = 1+1 = 2 \qquad u(1) = 1+1 = 2$

$\therefore \int_{-1}^{1} \frac{x^3 dx}{1+x^4} = \int_{2}^{2} \frac{1}{u} \frac{du}{4} = 0$

ii) $f(x) = \frac{x^3}{1+x^4}$ is odd function and continuous $\Rightarrow \left( f(-x) = \frac{(-x)^3}{1+(-x)^4} = -\frac{x^3}{1+x^4} = -f(x) \right)$

$\int_{-1}^{1} \frac{x^3}{1+x^4} dx = 0$

c) $\int_0^{-1} \frac{2x}{1+x^4} dx$

Let $u = x^2 \quad du = 2x\,dx \qquad u(0) = 0 \quad u(-1) = 1$

$$\int_0^{-1} \frac{2x\,dx}{1+x^4} = \int_0^1 \frac{du}{1+u^2} = \tan^{-1}(u)\Big|_0^1 = \tan^{-1}(1) - \tan^{-1}(0) = \frac{\pi}{4} - 0 = \frac{\pi}{4}$$

d) $\int_{-2}^{2} |x^2 - 1| dx$

$f(x) = |x^2-1|$ is even and continuous $\left( f(-x) = |(-x)^2 - 1| = |x^2-1| = f(x) \right)$

$$\therefore \int_{-2}^{2} |x^2-1|\,dx = 2\int_0^2 |x^2-1|\,dx = 2\left(\int_0^2 |(x-1)(x+1)|\,dx\right)$$

$$|x^2-1| = |(x-1)(x+1)| = \begin{cases} x^2-1 & 2 \geq x \geq 1 \\ 1-x^2 & 0 \leq x \leq 1 \end{cases}$$

$$\therefore \int_{-2}^{2} |x^2-1|\,dx = 2\int_0^2 |x^2-1|\,dx = 2\left(\int_0^1 (1-x^2)\,dx + \int_1^2 (x^2-1)\,dx\right)$$

$$= 2\left[\left(x - \frac{x^3}{3}\right)\Big|_0^1 + \left(\frac{x^3}{3} - x\right)\Big|_1^2\right]$$

$$= 2\left[\left(1 - \frac{1}{3}\right) + \left(\frac{8}{3} - 2\right) - \left(\frac{1}{3} - 1\right)\right]$$

$$= 2\left(\frac{2}{3} + \frac{2}{3} + \frac{2}{3}\right) = 4$$

4 **(10 pts.)** Prove that there is one and only one real root for the equation $2x - 1 - \sin x = 0$.

① Let $y = 2x - 1 - \sin x$ continuous

when $x$ is large, $y$ is positive (e.g. $y(\frac{\pi}{2}) = \pi - 1 - \sin\frac{\pi}{2} = \pi - 2 > 0$)

when $x$ is small, $y$ is negative (e.g. $y(0) = 0 - 1 - \sin 0 = -1 < 0$)

$\therefore$ by intermediate value theorem, there exists a number $c \in (0, \frac{\pi}{2})$ s.t. $y = 0$ at $c$.

② Now we want to prove that ~~there is~~ $c$ is the only root.

We use contradiction, if not, there exist $a$ and $b$ to be roots.

$y(a) = y(b) = 0$ and $a < b$

$\because$ $y$ is continuous and differentiable on $[a, b]$

$\therefore$ By Rolle's theorem (Mean Value Theorem)

we have some $d \in (a, b)$ s.t. $y'(d) = 0$

However, we know $y' = 2 - \cos x \geq 1$

We get contradiction, which completes the proof.

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2nd method for ②

Since $y'(x) \geq 1 \quad \forall x$

$y(c) = 0$

$\therefore$ $\forall x > c$ we have by FTC 2, $y(x) - y(c) = \int_c^x y'(x) \geq \int_c^x dx = x - c > 0$

$\therefore$ $y(x) > 0$

$\forall x < c$ we have $y(c) - y(x) = \int_x^c y'(x)dx \geq \int_x^c dx = c - x > 0$

$\therefore$ $y(x) < 0$

$\therefore$ No other solutions

**5 (8+8+4=20 pts.)** Let $f(x) = e^{2x} + e^x$.

i) Show that $f(x)$ is one to one.

ii) Find the inverse function $f^{-1}(x)$ (state domain explicitly).

iii) What is the range of $f^{-1}(x)$?

i) if not, $\exists\ x_1, x_2$ s.t. $f(x_1) = f(x_2)$, $x_1 < x_2$

Rolle's theorem $\Rightarrow \exists\ x \in (x_1, x_2)$ s.t. $f'(x) = 0$

However, $f'(x) = 2e^{2x} + e^x > 0$, contradiction.

ii) $y = f(x) = e^{2x} + e^x = (e^x)^2 + e^x = (e^x + \frac{1}{2})^2 - \frac{1}{4}$

$\Rightarrow y + \frac{1}{4} = (e^x + \frac{1}{2})^2 \Rightarrow e^x + \frac{1}{2} = \sqrt{y + \frac{1}{4}} \Rightarrow x = \ln\left(\sqrt{y + \frac{1}{4}} - \frac{1}{2}\right)$

$\therefore\ f^{-1}(x) = \ln\left(\sqrt{x + \frac{1}{4}} - \frac{1}{2}\right)$

the domain of $f^{-1}$ is the range of $f$.

by i) $f'(x) > 0 \Rightarrow$ increasing, $\lim_{x \to -\infty} f(x) = 0$, $\lim_{x \to \infty} f(x) = \infty$

$\therefore$ domain of $f^{-1}$ = range of $f$ $= (0, \infty)$

iii) range of $f^{-1}$ = domain of $f$ $= (-\infty, \infty)$

**6 (20 pts.)** Let $g(x) = \int_0^{x^3} \cos t\,dt$,

(a) Find the maximum possible value of $\frac{g'(x)}{x^2}$ for $x > 0$.

(b) Find the absolute maximum value of $g(x)$ in the interval $[0, (\frac{\pi}{2})^{1/3}]$.

a) $\cancel{g'(x) = 3x^2 \cos x^3}$ $\quad \cancel{g(x) = \sin x^3}$

$$g(x) = \int_0^{x^3} \cos t\,dt = \sin t \Big|_0^{x^3} = \sin x^3 - \sin 0 = \sin x^3$$

$$\therefore\ g'(x) = \cos(x^3) \cdot \frac{d}{dx} x^3 = 3x^2 \cos x^3$$

(alternatively, Let $u = x^3$, $g'(x) = \frac{dg}{du}\frac{du}{dx} = \cos(u) \cdot 3x^2 = 3x^2 \cos x^3$)

$$\therefore\ \frac{g'(x)}{x^2} = 3\cos x^3$$

we know $\cos(x^3)$ has max value to be 1.

$\therefore$ max of $\frac{g'(x)}{x^2}$ is 3.

b) i) critical points in $[0, (\frac{\pi}{2})^{1/3}]$: $g'(x) = 0 \Rightarrow x^2 \cos x^3 = 0$

$$\rightarrow x = 0, \left(\frac{\pi}{2}\right)^{1/3}$$

$$g(0) = \int_0^0 \cos t\,dt = 0$$

$$g\left((\pi/2)^{1/3}\right) = \int_0^{\pi/2} \cos t\,dt = \sin t \Big|_0^{\pi/2} = 1$$

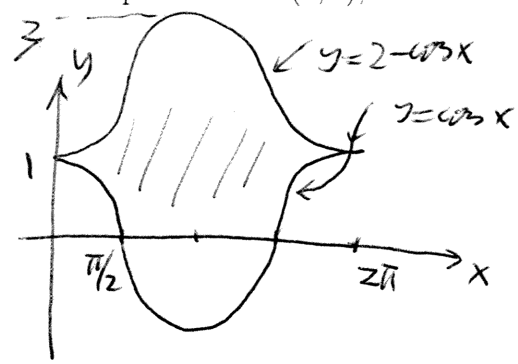
ii) end points are just critical points

$\therefore$ global max. value is $g\left((\pi/2)^{1/3}\right) = 1$.

**7 (20 pts.)** Sketch the regions and find the areas.

a) between $y = \cos x$ and $y = 2 - \cos x$, for $0 \le x \le 2\pi$.

b) the region bounded by the parabola $y = x^2$, the tangent line to this parabola at $(1, 1)$, and the $x$-axis.




a) From graph, we see $2-\cos x \ge \cos x$

∴ ~~graph~~ area of region

$$A = \int_0^{2\pi} \left|(2-\cos x) - \cos x\right| dx = \int_0^{2\pi} (2 - 2\cos x)\, dx$$

$$= (2x - 2\sin x)\Big|_0^{2\pi}$$

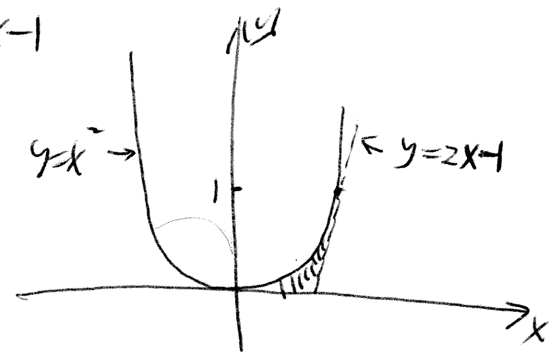
$$= (2\cdot 2\pi - 2\sin 2\pi) - (2\cdot 0 - 2\sin 0)$$

$$= 4\pi$$

b) $y = x^2$, $y' = 2x \Rightarrow$ at $(1,1)$ slope $y' = 2\cdot 1 = 2$

tangent line: $(y-1) = 2(x-1) \Rightarrow y = 2x-1$

express $x$ as function of $y$ (for $x, y \ge 0$)




∴ $x = \sqrt{y}$, $x = \frac{y+1}{2}$

$\frac{y+1}{2} \ge \sqrt{y}$ $\quad 0 \le y \le 1$

$$\therefore\ A = \int_0^1 \left(\frac{y+1}{2} - \sqrt{y}\right) dy = \left[\frac{1}{2}\left(\frac{y^2}{2} + y\right) - \frac{y^{3/2}}{3/2}\right]\Bigg|_0^1 = \frac{1}{2}\left(\frac{1}{2}+1\right) - \frac{1}{3/2}$$

$$= \frac{3}{4} - \frac{2}{3} = \frac{1}{12}$$

Alternatively, use $x$, $y = 2x-1$ intersect $x$-axis at $x = \frac{1}{2}$.

$$A = \int_0^{\frac{1}{2}} x^2\, dx + \int_{\frac{1}{2}}^1 \left[x^2 - (2x-1)\right] dx = \frac{x^3}{3}\Big|_0^{\frac{1}{2}} + \left(\frac{x^3}{3} - x^2 + x\right)\Big|_{\frac{1}{2}}^1$$

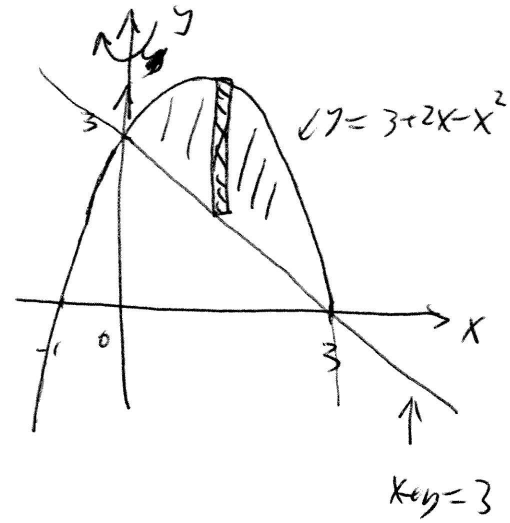
$$= \frac{1}{24} + \left(\frac{1}{3} - 1 + 1\right) - \left(\frac{1}{3}\cdot\frac{1}{8} - \frac{1}{4} + \frac{1}{2}\right)$$

$$= \frac{1}{24} + \frac{1}{3} - \frac{1}{24} - \frac{1}{4} = \frac{1}{12}$$

**8 (20 pts.)** Find the two volumes generated by rotating the regions bounded by the given curves about the specified axes. Sketch the regions.

(a) $y = 3 + 2x - x^2$ and $x + y = 3$, about $y$-axis,

(b) $y = x^2$ and $x = y^2$, about $y = -1$.




a) $y = 3+2x-x^2 = -(x^2-2x-3) = -(x-3)(x+1)$

$x+y=3 \Rightarrow y = 3-x$

points of intersection $3+2x-x^2 = 3-x \Rightarrow x^2-3x=0 \Rightarrow x=0,3$

$(0,3)$, $(3,0)$

typical shell: radius $x$ $(0 \le x \le 3)$

height $3+2x-x^2-(3-x) = 3x-x^2$

thick $dx$

$\therefore V = \int_0^3 2\pi x(3x-x^2)\,dx = \int_0^3 2\pi(3x^2-x^3)\,dx = 2\pi\left(x^3-\frac{x^4}{4}\right)\Big|_0^3 = 2\pi\left(27-\frac{81}{4}\right)$

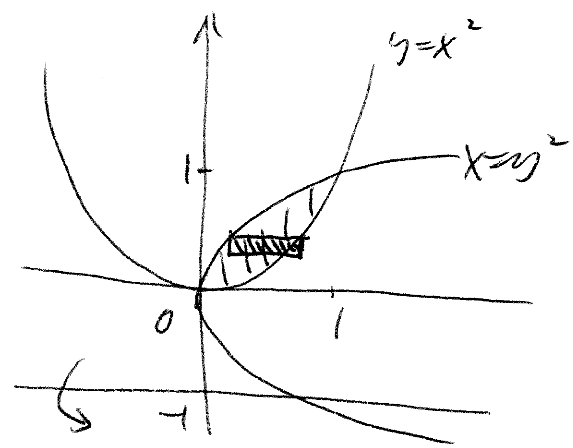
$= \frac{27}{2}\pi$

b) from graph, region in $x, y \ge 0$.

curves $y = x^2$, $y=\sqrt{x}$ $(x=y^2)$

points of intersection $x^2 = \sqrt{x} \Rightarrow x = 0, 1$

$(0,0)$, $(1,1)$




typical shell: radius $1+y$ $(0 \le y \le 1)$

height $\sqrt{y}-y^2$

thick $dy$

$\rightarrow V = \int_0^1 2\pi(1+y)(\sqrt{y}-y^2)\,dy = 2\pi\int_0^1 \left(y^{\frac{1}{2}} - y^2 + y^{\frac{3}{2}} - y^3\right)dy$

$= 2\pi\left(\boxed{\frac{1}{1/2}} - \frac{1}{3} + \frac{1}{5/2} - \frac{1}{4}\right) = 2\pi\left(2+\frac{2}{5}-\frac{1}{3}-\frac{1}{4}\right)$

$\frac{1}{3/2}$ 3 29

$= 2\pi\left(\frac{12}{5}-\frac{7}{12}\right) = \frac{109}{30}\pi$

Alternatively washer method

$r_{in} = x^2+1$, $r_{out} = \sqrt{x}+1$

3/2 $\frac{2}{5}+\frac{1}{12}$ 29

$V = \int_0^1 \pi\left[(\sqrt{x}+1)^2 - (x^2+1)^2\right]dx = \pi\int_0^1 (x+2\sqrt{x}-x^4-2x^2)\,dx = \pi\left(\frac{1}{2}+2\cdot\frac{1}{\frac{3}{2}}-\frac{1}{5}-2\cdot\frac{1}{3}\right) = \frac{109}{30}\pi$