Practice Midterm I
110.108 Calculus I for Engineers Fall 2010
Solutions

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1. (a) True
(b) False: $f(x) = |x|$ is not differentiable at $(0,0)$.
(c) False: the product rule is $[f(x)g(x)]' = f'(x)g(x) + f(x)g'(x)$. Lots of counterexamples
(d) True
(e) True

2. (a) $\lim_{x\to a} f(x) = L$ means that for all $\varepsilon > 0$, there exists $\delta > 0$ such that if $0 < |x - a| < \delta$, then $|f(x) - L| < \varepsilon$.
(b) We first find our δ. Take $L = 17$, $f(x) = 4x + 5$, and $a = 3$. If $|f(x) - L| < \varepsilon$, then

$$|4x + 5 - 17| = |4x - 12| = 4\,|x - 3| = 4\delta < \varepsilon.$$

Hence we can take $\delta < \frac{\varepsilon}{4}$. Now, let $\varepsilon > 0$ and $\delta < \frac{\varepsilon}{4}$. If $0 < |x - 3| < \delta$, then

$$|4x + 5 - 17| = |4x - 12| = 4\,|x - 3| < 4\delta < 4 \cdot \frac{\varepsilon}{4} = \varepsilon$$

and the proof is complete.

3. (a) We multiply the numerator and denominator by the conjugate to get

$$\begin{aligned}\lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} &= \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} = \lim_{h\to 0} \frac{x + h - x}{h\left(\sqrt{x+h} + \sqrt{x}\right)} \\ &= \lim_{h\to 0} \frac{h}{h\left(\sqrt{x+h} + \sqrt{x}\right)} = \lim_{h\to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{\sqrt{x} + \sqrt{x}} \\ &= \frac{1}{2\sqrt{x}}.\end{aligned}$$

(b) We have that

$$\begin{aligned}\lim_{x\to\infty} x - \sqrt{x+a}\sqrt{x+b} &= \lim_{x\to\infty} \frac{\left(x - \sqrt{x+a}\sqrt{x+b}\right)\left(x + \sqrt{x+a}\sqrt{x+b}\right)}{x + \sqrt{x+a}\sqrt{x+b}} \\ &= \lim_{x\to\infty} \frac{x^2 - (x+a)(x+b)}{x + \sqrt{x+a}\sqrt{x+b}} \\ &= \lim_{x\to\infty} \frac{-(a+b)x}{x + \sqrt{x+a}\sqrt{x+b}} - \frac{ab}{x + \sqrt{x+a}\sqrt{x+b}} \\ &= \lim_{x\to\infty} -(a+b) \cdot \frac{1}{1 + \frac{\sqrt{x+a}}{\sqrt{x}}\frac{\sqrt{x+b}}{\sqrt{x}}} \\ &= \lim_{x\to\infty} -(a+b) \cdot \frac{1}{1 + \sqrt{1 + \frac{a}{x}}\sqrt{1 + \frac{b}{x}}} \\ &= -\frac{a+b}{2}\end{aligned}$$

4. (a) We say that f is continuous at a if $\lim_{x\to a} f(x) = f(a)$.
(b) We require that $\lim_{x\to 2} f(x) = f(2)$. We're told that $f(2) = k$ so we need

$$\lim_{x\to 2} \frac{\sqrt{2x+5} - \sqrt{x+7}}{x - 2} = k.$$

This limit is equal to

$$\begin{aligned}\lim_{x\to 2}\frac{\sqrt{2x+5}-\sqrt{x+7}}{x-2} &= \lim_{x\to 2}\frac{\sqrt{2x+5}-\sqrt{x+7}}{x-2}\cdot\frac{\sqrt{2x+5}+\sqrt{x+7}}{\sqrt{2x+5}+\sqrt{x+7}}\\ &= \lim_{x\to 2}\frac{2x+5-x-7}{(x-2)\sqrt{2x+5}+\sqrt{x+7}}\\ &= \lim_{x\to 2}\frac{x-2}{(x-2)\sqrt{2x+5}+\sqrt{x+7}}\\ &= \lim_{x\to 2}\frac{1}{\sqrt{2x+5}+\sqrt{x+7}} = \frac{1}{3+3} = \frac{1}{6} = k.\end{aligned}$$

Hence $k=6$ gives us continuity.

5. (a) The definition of the derivative is

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}.$$

(b) Take $f(x) = \frac{x^2}{\sqrt{x+2}}$. Then using part (a),

$$\begin{aligned}f'(x) &= \lim_{h\to 0}\frac{\frac{(x+h)^2}{\sqrt{x+h+2}}-\frac{x^2}{\sqrt{x+2}}}{h} = \lim_{h\to 0}\frac{(x+h)^2\sqrt{x+2}-x^2\sqrt{x+h+2}}{h\sqrt{x+h+2}\sqrt{x+2}}\\ &= \lim_{h\to 0}\frac{(2xh+h^2)\sqrt{x+2}}{h\sqrt{x+h+2}\sqrt{x+2}} + \lim_{h\to 0}\frac{x^2\sqrt{x+2}-x^2\sqrt{x+h+2}}{h\sqrt{x+h+2}\sqrt{x+2}}\\ &= \lim_{h\to 0}\frac{(2x+h)\sqrt{x+2}}{\sqrt{x+h+2}\sqrt{x+2}} + x^2\lim_{h\to 0}\frac{\sqrt{x+2}-\sqrt{x+h+2}}{h\sqrt{x+h+2}\sqrt{x+2}}\\ &= \frac{2x}{\sqrt{x+2}} + x^2\lim_{h\to 0}\frac{\sqrt{x+2}-\sqrt{x+h+2}}{h\sqrt{x+h+2}\sqrt{x+2}}\cdot\frac{\sqrt{x+2}+\sqrt{x+h+2}}{\sqrt{x+2}+\sqrt{x+h+2}}\\ &= \frac{2x}{\sqrt{x+2}} + x^2\lim_{h\to 0}\frac{x+2-x-h-2}{h\sqrt{x+h+2}\sqrt{x+2}\left(\sqrt{x+2}+\sqrt{x+h+2}\right)}\\ &= \frac{2x}{\sqrt{x+2}} - x^2\lim_{h\to 0}\frac{1}{\sqrt{x+h+2}\sqrt{x+2}\left(\sqrt{x+2}+\sqrt{x+h+2}\right)}\\ &= \frac{2x}{\sqrt{x+2}} - \frac{x^2}{2\sqrt{x+2}^3}\ \text{(Whew!)}\end{aligned}$$

Note how we split the limit into two pieces, one piece which contained an h that we could factor and another piece which we had to multiply by the conjugate to factor out the h.

(c) We use the quotient rule, coupled with the power rule and chain rule. The quotient rule says

$$\left(\frac{h}{g}\right)' = \frac{h'g-hg'}{g^2}$$

so taking $h=x^2$ and $g=\sqrt{x+2}$, we get

$$f'(x) = \frac{(x^2)'\sqrt{x+2}-x^2\left(\sqrt{x+2}\right)'}{x+2}.$$

By the power rule, the derivative of $x^2 = 2x$ and by the chain rule, the derivative of $\sqrt{x+2}$ is $\frac{1}{2}(x+2)^{-1/2}$. Plugging this into the above, we get that

$$f'(x) = \frac{2x\sqrt{x+2}-\frac{x^2}{2\sqrt{x+2}}}{x+2} = \frac{2x}{\sqrt{x+2}} - \frac{x^2}{\sqrt{x+2}^3},$$

which agrees with what we found in part (b).

6. The hint turns this problem into an exercise in using the chain rule. We want

$$\frac{d}{dx}\left(x^x\right) = \frac{d}{dx}\left(e^{x\ln(x)}\right).$$

We write $u(x) = x\ln(x)$, so that we are left with $\frac{d}{dx}\left(e^{u(x)}\right)$. By the chain rule (and product rule), this is equal to

$$\frac{d}{dx}\left(e^{u(x)}\right) = \frac{du}{dx}e^{u(x)} = \left(x\left(\ln(x)\right)' + (x)'\ln(x)\right)e^{u(x)} = \left(\frac{x}{x} + 1\cdot\ln(x)\right)e^{u(x)} = \left(1+\ln(x)\right)e^{u(x)}.$$

But $e^{u(x)}$ is, by construction of $u(x)$, just equal to x^x. Hence $\frac{d}{dx}\left(x^x\right) = \left(1+\ln(x)\right)x^x$.

7. (a) Note that this is a function in x, not n so differentiating with respect to x simply requires the chain rule:

$$f'(x) = \lim_{n\to\infty} n\left(1+\frac{x}{n}\right)^{n-1}\cdot\frac{1}{n} = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^{n-1}.$$

To evaluate this limit, we rewrite it as

$$\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^{n-1} = \frac{\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^{n}}{\lim_{n\to\infty}\left(1+\frac{x}{n}\right)} = \frac{f(x)}{1+0} = f(x).$$

Hence $f'(x) = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = f(x)$.

(b) From part (a), $f'(x) = f(x)$ and we know that this means $f(x) = e^x$. Alternatively, you might recall that e^x can be *defined* as the function in part (a) and since we know the derivative of e^x is itself, we can guess that the derivative would have to be the same thing. Of course, you would still need to evaluate it as we did.