Calculus 108, Fall 2014 : Pre-Midterm Practice II

Johns Hopkins University

Problem 1

The sine, cosine hyperbolic functions are defined as

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

$$\cosh x = \frac{e^x + e^{-x}}{2}.$$

Prove the following identities on hyperbolic functions (first try without looking into the textbook or the class notes)

- (A) $\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y$
- (B) $\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y$.

Use this opportunity to recall also the expressions of inverse hyperbolic functions, their domains of definitions and their derivatives.

Problem 2

Using the closed interval method, find the absolute maximum and minimum values of the following functions in the given interval

- $x^5 - 5x^4 + 5x^3 + 1$, $[-1, 2]$
- $\sin 2x - x$, $[-\frac{\pi}{2}, \frac{\pi}{2}]$
- $\dfrac{1 - x + x^2}{1 + x - x^2}$, $[0, 1]$.

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Problem 3

- Check if the following function satisfies the conditions of the Rolle's mean value theorem in the interval $[-1, 1]$
$$f(x) = \frac{2 - x^2}{x^4}.$$

- Given,
$$f(x) = (x-1)(x-2)(x-3)(x-4),$$
without finding the derivative of the function, find the number of roots of the equation $f'(x) = 0$ and specify the intervals in which the roots lie.

Problem 4

- Comment on the concavity of the function $f(x) = x^5 - 5\,x^3 - 15\,x^2 + 30$ in the neighborhoods of each of the points $(3, 3)$ and $(1, 11)$.

- Find the points of inflection of the function $h(x) = x + 36\,x^2 - 2x^3 - x^4$.

Problem 5

Using the second derivative test, find the extreme values of

- $f(x) = x + \dfrac{a^2}{x}, (a > 0)$
- $g(x) = x + \sqrt{1 - x}\,.$

Problem 6

Using L'Hospital's rule find the following limits. Start by commenting on the type of indeterminate form of the initial limit.

- $\lim\limits_{x \to 0} \cot x - \dfrac{1}{x}$
- $\lim\limits_{x \to 0} \dfrac{a^x - b^x}{x\sqrt{1 - x^2}}$
- $\lim\limits_{x \to 0} (e^x + x)^{\frac{1}{x}}$ (hint: define $y = (e^x + x)^{\frac{1}{x}}$ and apply $\log \ldots$).

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Problem 7

- Show that $f(x) = (e^x + e^{-x})^2$ and $g(x) = (e^x - e^{-x})^2$ differ by a constant by proving that they are the antiderivatives of the same function.
- Find $g(u)$ if $g'(u) = \frac{u^2+\sqrt{u}}{u}$ and $g(1) = 3$.

Problem 8

Find the antiderivatives of

- $\dfrac{(1-x)^2}{x\sqrt{x}}$ and
- $\tan^2(x)$.

Problem 9

Using Riemann sums find the integral $\int_0^2 (x^3 + 1)d\,x$.

Some General Comments

It is perhaps needless to say that you should also review all the definitions, concepts, example problems discussed in the class, home works etc. Some basic topics are listed below with references to the textbook.

Definitions:

- Hyperbolic functions, inverse hyperbolic functions and their derivatives (§3.11)
- Critical number (§4.1, Def. 6)
- Inflection point (§4.3, p. 294)
- Antiderivative (§4.9, p. 344)

Theorems:

- Extreme Value Theorem (§4.1, Th. 3)
- Rolle's Theorem (§4.2, p. 284)
- Mean Value Theorem (§4.2, p. 285)

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- First Derivative Test (§4.3, p. 291)
- Second Derivative Test (§4.3, p. 295)

Furthermore, it may be helpful to remember some basic things like:

- Values of some trigonometric functions like $\sin 0 = \cos\frac{\pi}{2} = 0$, $\sin\frac{\pi}{6} = \cos\frac{\pi}{3} = \frac{1}{2}$, $\sin\frac{\pi}{3} = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$, $\sin\frac{\pi}{4} = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$. This will help you to obtain values of sine and cosine functions at some other points by exploiting periodicity, odd/even properties of trigonometric functions. Correspondingly you can also find the values of inverse trigonometric functions at those points.
- The following summation identities can also be handy

$$1 + 2 + 3 + 4 + \cdots + n = \Sigma_{i=1}^{n} i = \frac{n(n+1)}{2} \tag{1}$$

$$1^2 + 2^2 + 3^2 + 4^2 + \cdots + n^2 = \Sigma_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \tag{2}$$

$$1^3 + 2^3 + 3^3 + 4^3 + \cdots + n^3 = \Sigma_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4} \tag{3}$$

Finally, let me remind you what I already mentioned in the class. Learning mathematics is based on two complementary approaches: (1) Understanding the theory/reasoning behind the concepts, (2) Solving a good number of problems which help you become familiar the concepts. It is best if you prepare with this balance well before the exams.

Good luck for the exam,

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