110.108 CALCULUS 1
FALL 2013
FINAL EXAM

Name: ____

Recitation section:

___ 1. Tuesday 1:30 (V. Allard)
___ 2. Tuesday 3:00 (J. Jun)
___ 4. Thursday 4:30 (D. Ginsberg)
___ 5. Thursday 3:00 (D. Ginsberg)

Work quickly and carefully, and write your solutions clearly. Please show your work; partial credit will be given generously.

Statement of ethics
I agree to complete this exam without unauthorized assistance from any person, materials, or device.

Signature: ______________________________ Date: ______________

Problem	Score
1	/15
2	/15
3	/10
4	/10
5	/10
6	/15
7	/10
8	/10
TOTAL	/95

Problem 1. [15 points] Compute the following limits.

(a) $\displaystyle\lim_{x \to 0} \frac{\sin x}{\sec x}$

(b) $\displaystyle\lim_{t \to 1} \frac{\cos(\frac{\pi}{2} t)}{\ln t}$.

(c) $\displaystyle\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^{2x}$

(d) $\lim_{x \to 1} \left(\frac{1}{x-1} \int_1^x \sqrt{\tan^{-1} t}\, dt \right)$

(e) $\lim_{x \to 0^-} \frac{e^{1/x}}{x}$

(HINT: A change of variables may be helpful. This one is tricky, so don't burn too much time on it until you've worked the rest of the exam.)

Problem 2. [15 points] Compute the following.

(a) $\displaystyle\int \sec x \tan x \, dx$

(b) $\displaystyle\int_1^2 \left(x^2 - \frac{1}{x^2}\right) dx$

(c) $\displaystyle\int_1^3 \frac{e^{-1/t^2}}{t^3}\,dt$

(d) An equation for the tangent line to the curve $x^{1/2} + y^{1/2} = 3$ at the point $(4, 1)$.

(e) The amount of time (in years) it takes for your money in an account to triple if it's continuously compounded at an annual rate of $r = 0.05$.

Problem 3. [10 points] Find $\lim_{x \to 2} \frac{2x^2 - 12x + 16}{x - 2}$ and give an ϵ-δ proof that you're right.

Problem 4. [$5+5=10$ points] Let R be the region in the first quadrant bounded by the ellipse

$$\frac{x^2}{4}+\frac{y^2}{9}=1$$

and the x- and y-axes. Using any method you like, find the volumes obtained by rotating R (a) around the x-axis, and (b) around the y-axis. (HINT: It's up to you to figure out the limits of integration.)

Problem 5. $[4+6=10$ points$]$ An evil mastermind with a keen sense of mathematical aesthetics and a cheesy sense of drama has laid out a fuse along a portion of the curve $y = \ln x$, $x \geq 1$, which leads to a pile of TNT.

(a) Set up, but DO NOT EVALUATE, an integral for the length $L(s)$ of the fuse from $x = 1$ to $x = s$.

(b) Once the fuse is lit, it burns at a constant rate of 1 cm/sec. What's the rate of change of s when $s = 10$ cm? (HINT: $\frac{dL}{dt} = \frac{dL}{ds}\frac{ds}{dt}$.)

Problem 6. $[5 + 10 = 15$ points$]$

(a) Set up and evaluate an integral to show that a right circular cone of height h and base radius r has surface area $\pi r\sqrt{r^2 + h^2}$. (No points for solving this problem without calculus. To be clear, you should regard the base of the cone as "open," i.e. it does not contribute any surface area.)

(b) You are planning to throw a surprise birthday party for a certain professor next week, and you want to make party hats in the shape of a right circular cone. To make room for how much your brain has grown this semester, the hats should have as much volume as possible. If each hat is to be made from 30π square inches of paper, what should the height and base radius be? As usual, your work must justify your answer. (HINT: As a present to you, the volume of a right circular cone with height h and base radius r is $\frac{1}{3}\pi r^2 h$. Also, it is equivalent to find dimensions that maximize the volume-squared.)

Problem 7. [$5 + 5 = 10$ points]

(a) Let f and g be functions defined on an open interval containing a. Define "derivative of f at a" and "g is continuous at a."

(b) Let

$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0. \end{cases}$$

Compute $f'(x)$ for all $x \in \mathbb{R}$. (HINT: Consider $x = 0$ and $x \neq 0$ separately.) Is f' continuous at 0? Explain.

Problem 8. [10 points] Suppose that f is integrable on $[a, b]$, and let m and M be constants such that $m \leq f(x) \leq M$ for all $x \in [a, b]$. Using theorems from lecture or the text, prove that the average value of f on $[a, b]$ is between m and M. (HINT: A good way to get started is to write down how the average value of f is defined.)