Solutions Midterm Exam 1 — Oct. 7, 2015

1. (20 points) Let $f(x) = 2x^2 + 4x - 1$. Determine the largest value $R > 0$ so that f is one-to-one on the interval $(-R, R)$. Determine f^{-1}, the inverse of f on this interval. What is the domain of f^{-1}?

We re-write f as

$$f(x) = 2(x+1)^2 - 3$$

and observe that this is the translation and vertical stretch of the function $y = x^2$. In particular, we see that the function is decreasing on $(-\infty, -1]$ and increasing on $[-1, \infty)$ and so the largest open interval on which the function is one-to-one is $(-1, 1)$.

To determine the inverse function we solve

$$y = 2(x+1)^2 - 3$$

for x. Doing so, we see that

$$x = -1 \pm \sqrt{\frac{1}{2}(y+3)}.$$

As x has to lie in $(-1, 1)$ we conclude that

$$f^{-1}(y) = -1 + \sqrt{\frac{1}{2}(y+3)}.$$

Finally, as the domain of f^{-1} is the range of f, we see that the domain is $(f(-1), f(1)) = (-3, 5)$.

2. Evaluate the following limits. You may use any technique you like as long as you justify your steps.

   (a) (10 points) $\lim_{x\to 1} \cos\left(\pi \frac{\sqrt{x}-1}{x-1}\right)$.

As $\cos(\pi x)$ is continuous at all values x, we have

$$\lim_{x\to 1} \cos\left(\pi\left(\frac{\sqrt{x}-1}{x-1}\right)\right) = \cos\left(\pi\left(\lim_{x\to 1}\frac{\sqrt{x}-1}{x-1}\right)\right)$$

provided the second limit exists. Hence, as

$$\lim_{x\to 1}\frac{\sqrt{x}-1}{x-1} = \lim_{x\to 1}\left(\frac{\sqrt{x}-1}{x-1}\,\frac{\sqrt{x}+1}{\sqrt{x}+1}\right) = \lim_{x\to 1}\frac{1}{\sqrt{x}+1} = \frac{1}{2},$$

the limit exists and equals $\cos(\pi/2) = 0$.

(b) (10 points) $\lim_{x\to 0} \frac{\tan(2x)\sin(x)}{x^2}$.

Observe that if $f(x) = \tan(2x)$, then $f(0) = 0$. We then have

$$\lim_{x\to 0} \frac{\tan(2x)}{x} = \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = f'(0) = 2.$$

Where we used that $f'(x) = 2\sec^2(2x)$ so $f'(0) = 2$. Similarly, if $g(x) = \sin(x)$, then

$$\lim_{x\to 0} \frac{\sin(x)}{x} = \lim_{x\to 0} \frac{g(x) - g(0)}{x - 0} = g'(0) = \cos(0) = 1.$$

Hence, by the product law

$$\lim_{x\to 0} \frac{\tan(2x)\sin(x)}{x^2} = \left(\lim_{x\to 0} \frac{\tan(2x)}{x}\right)\left(\lim_{x\to 0} \frac{\sin(x)}{x}\right) = 2 * 1 = 2.$$

3. Give examples of functions with the given property. You do not need to justify your answers.

(a) (5 points) A jump discontinuity at $x = 2$.

$$f(x) = \begin{cases} x & x \geq 2 \\ 0 & x < 2 \end{cases}$$

(b) (5 points) An infinite discontinuity at $x = -1$.

$$f(x) = \frac{1}{(x+1)^2}$$

(c) (5 points) A removable discontinuity at $x = 0$.

$$f(x) = \begin{cases} x & x \neq 0 \\ 3 & x = 0 \end{cases}$$

(d) (5 points) A discontinuity at $x = 0$ which is not one of the preceding three types.

$$f(x) = \begin{cases} 0 & x = 0 \\ \sin\left(\frac{1}{x}\right) & x \neq 0 \end{cases}$$

4. Let $f(x) = x^3 - 2x^2 - x + 2$.

(a) (10 points) Determine the equation of the tangent line to $y = f(x)$ at $(a, f(a))$ when $a = 2$.

The slope of the tangent line at $(a, f(a))$ is given by $f'(a) = 3a^2 - 4a - 1$. Hence, the general equation of the tangent line is $y = (3a^2 - 4a - 1)(x - a) + a^3 - 2a^2 - a + 2$. When $a = 2$, this equation becomes $y = 3(x - 2)$.

(b) (10 points) Determine the value(s) a so the tangent line at $(a, f(a))$ is parallel to the line $y = -2x$.

The slope of the tangent line at $(a, f(a))$ is given by $f'(a) = 3a^2 - 4a - 1$. Hence, the tangent line is parallel to the line $y = -2x$ when and only when $3a^2 - 4a - 1 = -2$ that is $3a^2 - 4a + 1 = 0$. One can factor $(3a^2 - 4a + 1) = (3a - 1)(a - 1)$ and so conclude that a must be either 1 or 1/3.

5. (20 points) Let f be a function so that $f(1) = 1$, $f'(1) = -1$ and $f''(1) = 2$. If $h(x) = f(f(x))$ determine $h''(1)$.

Using the Chain Rule, we have that

$$h'(x) = f'(f(x))f'(x)$$

and so, using the Product rule and the Chain Rule again, we conclude that

$$h''(x) = f''(f(x))(f'(x))^2 + f'(f(x))f''(x)).$$

Hence, $h''(1) = f''(f(1))(f'(1))^2 + f'(f(1))f''(1) = 2(-1)^2 - 1 * 2 = 0$.