Name: ____ **Section Number:**__________

110.108 CALCULUS I (Physical Sciences & Engineering)
FALL 2011
MIDTERM EXAMINATION Solutions
November 30, 2011

Instructions: The exam is **7** pages long, including this title page and a spare sheet at the end. The number of points each problem is worth is listed after the problem number. The exam totals to one hundred points. For each item, please **show your work** or **explain** how you reached your solution. Please do all the work you wish graded on the exam. Good luck !!

PLEASE DO NOT WRITE ON THIS TABLE !!

Problem	Score	Points for the Problem
1		10
2		25
3		25
4		20
5		20
TOTAL		100

Statement of Ethics regarding this exam

I agree to complete this exam without unauthorized assistance from any person, materials, or device.

Signature: ____ **Date:**__________

Question 1. [10 points] Choose exactly ONE of the following two problems, and find the equation of the line tangent to the graph of the equation at the point specified:

(a) $y = x^{\sin \pi x}$ at the point $p = (1, 1)$.

Strategy: Here, we use logarithmic differentiation to find the derivative, evaluate at the point p, then write out the point-slope form of the equations of a line at p with the derivative as the slope.

Solution: First, note that at and near p, $y > 0$ since $y = 1$ at p. And whenever two sides of an equation are equal and positive, it follows that the natural log of each side is also equal. Hence

$$\ln y = \ln x^{\sin \pi x} = \sin \pi x \ln x.$$

Differentiating both sides gives us

$$\begin{aligned}
\frac{d}{dx}\Big[\ln y &= \sin \pi x \ln x\Big] \\
\frac{d}{dx}\Big[\ln y\Big] &= \frac{d}{dx}\Big[\sin \pi x \ln x\Big] \\
\frac{1}{y}\frac{dy}{dx} &= (\pi \cos \pi x)\ln x + \frac{\sin \pi x}{x} \\
\frac{dy}{dx} &= y\left(\pi \cos \pi x \ln x + \frac{\sin \pi x}{x}\right).
\end{aligned}$$

We could re-substitute the original function for y in this last equation, or leave it as it, noting that $p = (1, 1) = (x, y)$. And

$$\left.\frac{dy}{dx}\right|_{(1,1)} = 1\left(\pi(1)(0) + \frac{0}{1}\right) = 0.$$

Hence the equation of the line tangent to the graph of $y = x^{\sin \pi x}$ at $p = (1, 1)$, is

$$y - 1 = 0(x - 1) = 0 \quad \text{or} \quad y = 1.$$

(b) $xy - y^2x = x - 6$ at the point $p = (2, -1)$.

Strategy: Here, we use implicit differentiation to find the derivative, evaluate at the point p, then write out the point-slope form of the equations of a line at p with the derivative as the slope.

Solution: First, note that we cannot solve this equation for y as a function of x, so implicit differentiation is necessary. Hence we differentiate the entire equation with respect to x:

$$\frac{d}{dx}\Big[xy - y^2x = x - 6\Big].$$

Note that we will need to use the Product Rule on both of the terms in the left-hand side. We get

$$\begin{aligned}
\frac{d}{dx}\Big[xy - y^2x\Big] &= \frac{d}{dx}\Big[x - 6\Big] \\
y + x\frac{dy}{dx} - 2y\frac{dy}{dx}x - y^2 &= 1.
\end{aligned}$$

Solving for $\frac{dy}{dx}$, we get

$$\frac{dy}{dx} = \frac{1 - y + y^2}{x - 2xy} \quad \text{and} \quad \left.\frac{dy}{dx}\right|_{(2,-1)} = \frac{1 - (-1) + (-1)^2}{2 - 2(2)(-1)} = \frac{3}{6} = \frac{1}{2}.$$

Hence the equation for the line tangent to the graph of the equation at p is

$$y - (-1) = \frac{1}{2}(x - 2).$$

Question 2. [25 points] Given $f(x) = 3x^4 - 4x^3$, do the following:

(a) Find and classify ALL critical points.

Strategy: Here, $f(x)$ is a polynomial and hence is differentiable for all x. Hence all critical points will satisfy $f'(x) = 0$. We solve this equation and then use derivative tests to classify them.

Solution: First, we calculate $f'(x)$ and solve $f'(x) = 0$: $f'(x) = 12x^3 - 12x^2 = 12x^2(x-1)$, and $12x^2(x-1) = 0$ is solved by $x = 0$ and $x = 1$. We first use the second derivative test to classify these critical points. Here $f''(x) = 36x^2 - 24x$, and it is immediate that $f''(1) = 12 > 0$. Hence by the Second Derivative Test, $x = 1$ is a local minimum.

But $f''(0) = 0$, hence the Second Derivative Test fails to provide a classification of this critical point. But we also know the following: For small values of $x < 0$, $f'(x) = 12x^2(x-1) < 0$, and also for small values of $x > 0$, $f'(x) = 12x^2(x-1) < 0$. Hence by the First Derivative Test, the function is falling on both sides of $x = 0$. Hence $x = 0$ is neither a local min nor a local max (you can also check the values of $f(x)$ on both sides of $x = 0$ to verify this.)

(b) Determine the concavity of $f(x)$ and locate any and all inflection points.

Strategy: Here, the sign of $f''(x)$ determines the concavity, and where the sign changes is where the inflection points are located.

Solution: Here, $f''(x) = 36x^2 - 24x = 12x(3x-2)$, and $12x(3x-2) = 0$ is solved by $x = 0$ and $x = \frac{2}{3}$. Checking the sign directly on each interval determined by these two points, we see that

$$\begin{aligned} f''(x) > 0 &\quad \text{on} \quad (-\infty, 0)\,, \\ f''(x) < 0 &\quad \text{on} \quad \left(0, \frac{2}{3}\right)\,, \text{ and} \\ f''(x) > 0 &\quad \text{on} \quad \left(\frac{2}{3}, \infty\right)\,. \end{aligned}$$

Hence $f(x)$ is concave up on $(-\infty, 0) \cup \left(\frac{2}{3}, \infty\right)$ and concave down on $\left(0, \frac{2}{3}\right)$. And since both $x = 0$ and $x = \frac{2}{3}$, and the concavity is different on teach of the sides of these two points, the two points $x = 0$ and $x = \frac{2}{3}$ are inflection points.

(c) Find the global maximum of $f(x)$ on the interval $[-1, 1]$.

Strategy: Here, both critical points are in the interval. So we check the function values at these two critical points, and also at the one end point that is not critical. The maximum is the largest value of these three.

Solution: Here, $x = 0$ and $x = 1$ are the two critical points of $f(x)$ in the interval $[-1, 1]$. Hence we check the function values at these two points and also at the end point $x = -1$. $f(0) = 0$, $f(1) = 3(1)^4 - 4(1)^3 = -1$, and $f(-1) = 3(-1)^4 - 4(-1)^3 = 3 + 4 = 7.$. Hence the global maximum of $f(x)$ on $[-1, 1]$ is 7.

Question 3. [25 points] Evaluate the following:

(a) $\displaystyle\int \left(2^x - \frac{x-1}{x^2}\right)\,dx$

Strategy: Here, knowing the integral of a sum is the sum of the integrals, we simply break up the fraction into two fractions and integrate each of hte summands of the integrand separately.

Solution:

$$\int \left(2^x - \frac{x-1}{x^2}\right)dx = \int \left(2^x - \frac{1}{x} + \frac{1}{x^2}\right)dx = \int 2^x\,dx - \int \frac{1}{x}\,dx + \int \frac{1}{x^2}\,dx = \frac{2^x}{\ln 2} - \ln|x| - \frac{1}{x} + C.$$

(b) $\displaystyle\frac{d}{dx}\left(\int_{\sin(2x)+\sqrt{x+1}}^{3} \ln(t+e-1)\,dt\right)\Bigg|_{x=0}$

Strategy: First, we switch the limits to place the expression involving x at the top. Then we use the Chain Rule to calculate the derivative. Note that the act of differentiating a function defined via an integral is simply to take the integrand and insert the expression for x directly into every instance of the variable t. And since the expression for x is a function of x, we must "tack on" the derivative of this expression to the end as in the Chain Rule.

Solution: Note first that

$$\int_{\sin(2x)+\sqrt{x+1}}^{3} \ln(t+e-1)\,dt = -\int_{3}^{\sin(2x)+\sqrt{x+1}} \ln(t+e-1)\,dt.$$

Thus

$$\begin{aligned}\frac{d}{dx}\left(\int_{\sin(2x)+\sqrt{x+1}}^{3} \ln(t+e-1)\,dt\right) &= \frac{d}{dx}\left(-\int_{3}^{\sin(2x)+\sqrt{x+1}} \ln(t+e-1)\,dt\right)\\ &= -\ln\left(\sin(2x)+\sqrt{x+1}+e-1\right)\frac{d}{dx}\left[\sin(2x)+\sqrt{x+1}\right]\\ &= -\ln\left(\sin(2x)+\sqrt{x+1}+e-1\right)\left(2\cos(2x)+\frac{1}{2\sqrt{x+1}}\right).\end{aligned}$$

Evaluating this last expression at $x=0$, we get

$$\begin{aligned}&-\ln\left(\sin(2x)+\sqrt{x+1}+e-1\right)\left(2\cos(2x)+\frac{1}{2\sqrt{x+1}}\right)\Bigg|_{x=0}\\ &\qquad= -\ln\left(\sin(0)+\sqrt{0+1}+e-1\right)\left(2\cos(0)+\frac{1}{2\sqrt{0+1}}\right)\\ &\qquad= -(\ln e)\left(2+\frac{1}{2}\right) = -\frac{5}{2}.\end{aligned}$$

(c) $\displaystyle\int_0^2 \frac{x^3}{\sqrt{1+2x^2}}\,dx$

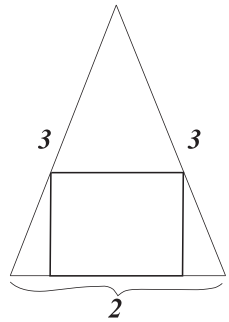
Strategy: We use the Substitution Rule here, where everything under the radical becomes our new variable.

Solution: Using the substitution $u = 1+2x^2$, where $du = 4x\,dx$, or $\frac{1}{4}\,du = x\,dx$, we note that there is still an x^2 in the integrand. We can address this by solving the original substitution expression for x^2, getting $x^2 = \frac{1}{2}(u-1)$. We can also switch the limits via the substitution: when $x=0$, $u = 1+2(0)^2 = 1$, and when

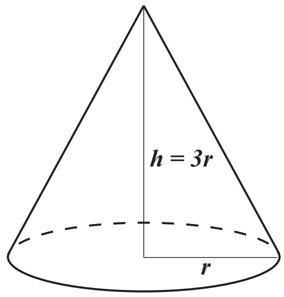
$x = 2$, $u = 1 + 2(2)^2 = 9$. Putting all of these into place, we get:

$$\begin{aligned}
\int_0^2 \frac{x^3}{\sqrt{1+2x^2}}\,dx &= \int_1^9 \frac{1}{4}\,\frac{\frac{1}{2}(u-1)}{\sqrt{u}}\,du \\
&= \frac{1}{8}\,\frac{(u-1)}{\sqrt{u}}\,du \\
&= \frac{1}{8}\left(u^{\frac{1}{2}} - u^{-\frac{1}{2}}\right)\,du \\
&= \frac{1}{8}\left(\frac{2}{3}u^{\frac{3}{2}} - 2u^{\frac{1}{2}}\right)\Big|_1^9 \\
&= \frac{1}{8}\left(\frac{2}{3}(9)^{\frac{3}{2}} - 2(9)^{\frac{1}{2}} - \frac{2}{3}(1)^{\frac{3}{2}} + 2(1)^{\frac{1}{2}}\right) \\
&= \frac{1}{8}\left(18 - 6 - \frac{2}{3} + 2\right) = \frac{40}{24} = \frac{5}{3}.
\end{aligned}$$

Question 4. [20 points] Find the dimensions of the largest rectangle that can be inscribed in an isosceles triangle with base 2 and other side lengths 3, when one of the edges of the rectangle rests on the base of the triangle.



Question 5. [20 points] Sand is being emptied from a hopper at a rate of 10 ft^3/min. The sand forms a conical pile whose height is always three times the radius of the base. At what rate is the radius of the base increasing when the height is 5 ft? (Hint: The volume of the cone is $V = \frac{1}{3}\pi r^2 h$.



Extra Page