Name: ______________________________ Section Number: ____________

110.201 Linear Algebra
FALL 2013
MIDTERM EXAMINATION
December 4, 2013

Instructions: The exam is **6** pages long, including this title page and a blank page at the end. **All pages must stay with the exam.** The number of points each problem is worth is listed after the problem number. The exam totals to one hundred points. For each item, please **show your work** or **explain** how you reached your solution. Please do all the work you wish graded on the exam. Good luck !!

PLEASE DO NOT WRITE ON THIS TABLE !!

Problem	Score	Points for the Problem
1		15
2		30
3		25
4		30
TOTAL		100

Statement of Ethics regarding this exam

I agree to complete this exam without unauthorized assistance from any person, materials, or device.

Signature: ______________________________ **Date:** ____________

PLEASE SHOW ALL WORK, EXPLAIN YOUR REASONS, AND STATE ALL THEOREMS YOU APPEAL TO

Question 1. [15 points] Given $\mathbf{A} = \begin{bmatrix} 1 & -5 \\ 2 & -5 \end{bmatrix}$, do the following:

(a) Calculate the matrix $\mathbf{S}$ so that $\mathbf{S}^{-1}\mathbf{A}\mathbf{S} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$, for $a, b \in \mathbb{R}$ and determine a and b.

(b) Write the linear transformation $\mathbf{S}^{-1}\mathbf{A}\mathbf{S}$ as a composition of a scaling and a rotation. What is the scaling factor in this case?

(a) We solve for the complex eigenvectors of A to produce S:

$$\det(A - \lambda I_2) = \lambda^2 + 4\lambda + 5 = 0 \Rightarrow \lambda = \frac{-4 \pm \sqrt{16-20}}{2}$$

$\lambda = -2 \pm i$

Let $\lambda = -2+i$, then $A\vec{v} = \lambda\vec{v} \Rightarrow$

$$v_1 - 5v_2 = (-2+i)v_1 \Rightarrow -5v_2 = (-3+i)v_1$$

choose $v_1 = -5$, $v_2 = (-3+i)$. then $\vec{v} = \begin{bmatrix} -5 \\ -3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} i$

So that $S = \begin{bmatrix} 0 & -5 \\ 1 & -3 \end{bmatrix}$, and $S^{-1}AS = \begin{bmatrix} -2 & -1 \\ 1 & -2 \end{bmatrix}$

(b) $S^{-1}AS = \begin{bmatrix} -2 & -1 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} \sqrt{5} & 0 \\ 0 & \sqrt{5} \end{bmatrix} \begin{bmatrix} -2/\sqrt{5} & -1/\sqrt{5} \\ 1/\sqrt{5} & -2/\sqrt{5} \end{bmatrix}$.

Scaling factor is $\sqrt{5}$.

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Question 2. [30 points] For $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 2 \\ -2 \\ 2 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 2 \end{bmatrix}$, let $V = \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2\}$. Do the following:

(a) Calculate the area (the 2-volume) of the parallelepiped defined by $\mathbf{v}_1$ and $\mathbf{v}_2$.

(b) Calculate the matrix of the orthogonal projection of $\mathbb{R}^4$ onto V. (*Hint: you will need to produce an orthonormal basis for V for this.*)

(c) Determine $\operatorname{proj}_V \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$.

(a) Let $A = \begin{bmatrix} | & | \\ \vec{v}_1 & \vec{v}_2 \\ | & | \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 0 \\ -2 & 0 \\ 2 & 2 \end{bmatrix}$. Then area is $\sqrt{\det[A^T A]} = \sqrt{\det\left(\begin{bmatrix} 2 & 2 & -2 & 2 \\ 2 & 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} 2 & 2 \\ 2 & 0 \\ -2 & 0 \\ 2 & 2 \end{bmatrix}\right)}$

$$= \sqrt{\det\begin{bmatrix} 16 & 8 \\ 8 & 8 \end{bmatrix}} = \sqrt{64} = 8$$

(b) Orthonormal basis: $\vec{u}_1 = \begin{bmatrix} 1/2 \\ 1/2 \\ -1/2 \\ 1/2 \end{bmatrix}$, $v_2^\perp = \vec{v}_2 - (\vec{u}_1 \cdot \vec{v}_2)\vec{u}_1$

$$= \begin{bmatrix} 2 \\ 0 \\ 0 \\ 2 \end{bmatrix} - (2)\begin{bmatrix} 1/2 \\ 1/2 \\ -1/2 \\ 1/2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix} \text{ so } \vec{u}_2 = \begin{bmatrix} 1/2 \\ -1/2 \\ 1/2 \\ 1/2 \end{bmatrix}$$

Project is QQ^T, where $Q = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \\ -1 & 1 \\ 1 & 1 \end{bmatrix}$, so

$$QQ^T = \frac{1}{4}\begin{bmatrix} 1 & 1 \\ 1 & -1 \\ -1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 2 & 0 & 0 & 2 \\ 0 & 2 & -2 & 0 \\ 0 & -2 & 2 & 0 \\ 2 & 0 & 0 & 2 \end{bmatrix}$$

(c) $$\frac{1}{4}\begin{bmatrix} 2 & 0 & 0 & 2 \\ 0 & 2 & -2 & 0 \\ 0 & -2 & 2 & 0 \\ 2 & 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 10 \\ -2 \\ 2 \\ 10 \end{bmatrix}$$

Question 3. [25 points] Let $W \subset P_4$ be the subset of polynomials of degree 4 or less that are even (that is, $f(-t) = f(t)$, $\forall t \in \mathbb{R}$). Do the following:

(a) Show W is a linear subspace of P_4.

(b) Given the basis $\{1, t^2, t^4\}$ for W, construct the matrix of the linear transformation $T: W \to W$ where $T(f(t)) = 3f''(t) - 2f(t)$.

a) Let $f(t)$, $g(t) \in W$.

check 1) $0 \in W$

2) $f(t) + g(t) \in W$

3) $kf(t) \in W$.

1) the neutral element $\in W$, because 0 is an even polynomial

ie. if $h(t) = 0 = 0t^4 + 0.t^2 + 0 \Rightarrow$

$h(-t) = 0(-t)^4 + 0(-t)^2 + 0 = 0.$

$\Rightarrow h(t) = h(-t) \Rightarrow h(t) = 0 \in W.$

2). let $h(t) = f(t) + g(t)$, then show $h(t) = h(-t)$.

$h(t) = f(t) + g(t)$

$h(-t) = f(-t) + g(-t)$ but $g, f \in W$ so.

$h(-t) = f(t) + g(t)$

$\Rightarrow h(t) = h(-t) \Rightarrow h(t) = g + f \in W.$

3) Let $h(t) = kf(t)$

show $h(t) = h(-t)$

$h(t) = kf(t)$

$h(-t) = kf(-t) = kf(t)$ b/c $f \in W$ & $f(t) = f(-t)$

$\Rightarrow kf(t) = h(t) \in W.$

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b) the matrix B of the transformation $T: W \to W$

is:

$$B = \left[[T(1)]_{\mathcal{B}} \quad [T(t^2)]_{\mathcal{B}} \quad [T(t^4)]_{\mathcal{B}} \right].$$

where $\mathcal{B} = \{1, t^2, t^4\}$ (the given basis for W.

$T(f(1)) = -2 = \underline{-2} \cdot 1 + \underline{0} \cdot t^2 + \underline{0} \cdot t^4$

$T(f(t^2)) = 3(t^2)'' - 2t^2 = 6 - 2t^2 = \underline{6} \cdot 1 - \underline{2} \cdot t^2 + \underline{0} \cdot t^4$

$T(f(t^4)) = 3(t^4) - 2t^4 = 36t^2 - 2t^4 = \underline{0} \cdot 1 + \underline{36} t^2 - \underline{2} t^4$

$$\Rightarrow B = \begin{bmatrix} -2 & 6 & 0 \\ 0 & -2 & 36 \\ 0 & 0 & -2 \end{bmatrix}$$

Question 4. [30 points] For $\mathbf{A} = \begin{bmatrix} -1 & -1 & 0 \\ 1 & -3 & 0 \\ -3 & 0 & 0 \end{bmatrix}$, do the following:

(a) Calculate the eigenvalues of **A** and determine their algebraic multiplicities.

(b) Determine the eigenspaces of **A** and the geometric multiplicities of the eigenvalues.

(c) Diagonalize **A**, if possible, or explain why it is not diagonalizable.

to compute λ-s, solve:

a) $\det(A - I\lambda) = 0$

$$\det \begin{bmatrix} -1-\lambda & -1 & 0 \\ 1 & -3-\lambda & 0 \\ -3 & 0 & -\lambda \end{bmatrix} = -\lambda \det \begin{bmatrix} -1-\lambda & -1 \\ 1 & -3-\lambda \end{bmatrix} =$$

$$= -\lambda\left((1+\lambda)(3+\lambda)+1\right) = -\lambda(\lambda+2)^2.$$

$\Rightarrow \lambda_1 = 0$ alg. multip. 1

$\lambda_2 = -2$ alg. multip. 2.

b) $E_{\lambda=0} = \ker(A - 0 \cdot I) = \ker(A) = \left\{ \vec{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \right\}.$

$$\begin{bmatrix} -1 & -1 & 0 \\ 1 & -3 & 0 \\ -3 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \left. \begin{array}{l} -x-y=0 \\ x-3y=0 \\ -3x=0 \end{array} \right\} \Rightarrow \begin{array}{l} x=0 \\ y=0 \\ z=t \in \mathbb{R} \end{array}$$

$$\vec{v} = \begin{bmatrix} 0 \\ 0 \\ t \end{bmatrix} \Rightarrow E_{\lambda=0} = \operatorname{span}\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

$E_{\lambda=-2} = \ker(A+2I) \Rightarrow$

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & -1 & 0 \\ -3 & 0 & +2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \left. \begin{aligned} x - y &= 0 \\ -3x + 2z &= 0 \end{aligned} \right\} \Rightarrow \begin{aligned} x &= y \\ z &= \tfrac{3}{2} x \end{aligned}$$

$$\Rightarrow E_{\lambda=-2} = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 3/2 \end{bmatrix} \right\}.$$

c) No, A is not diagonalizable because

$\left\{ \text{alg. multip. of } \lambda=-2 \text{ is } 2 \right\} \underset{\neq}{>} \left\{ \text{geom. multip. } \lambda=-2 \text{, which is } 1 \right\}$

equivallently, the sums of the geom. multiplicities for each λ sum up to 2 $\Rightarrow$ we can't form an eigen basis for $\mathbb{R}^3$.

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