LINEAR ALGEBRA (MATH 110.201)

FINAL EXAM - DECEMBER 2015

Name: ____________________

Section number/TA: ____________________

Instructions:

(1) Do not open this packet until instructed to do so.
(2) This midterm should be completed in **3 hours**.
(3) Notes, the textbook, and digital devices **are not permitted**.
(4) Discussion or collaboration is **not permitted**.
(5) All solutions must be written on the pages of this booklet.
(6) Justify your answers, and write clearly; points will be subtracted otherwise.

Exercise	Points	Your score
1	5	
2	5	
3	5	
4	5	
5	8	
6	8	
7	8	
8	8	
9	8	
10	10	
11	8	
12	12	
Total	90	

Exercise 1 (5 points): Let a, b be real numbers. Consider the following system of equations:

$$\begin{aligned} X + Y + 2Z &= a \\ 2X + 2Y + 3Z &= b \\ 3X + 3Y + 4Z &= a + b \end{aligned}$$

(1) Determine all possible values of a, b for which the above system has a solution. When the system has a solution, describe all solutions in terms of a and b.

(2) Are there any real numbers a, b for which the system of equations above has exactly one solution?

Solution:

Solution (continued):

Exercise 2 (5 points):

(1) Give an example of 2×2 matrices C and D such that $CD \neq DC$.

(2) Let $A = \left(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right)$ with a a real number. Show that if B is any 2×2 matrix, then $AB = BA$.

(3) Are there any other 2×2 matrices A having the property that $AB = BA$ for all 2×2 matrices B? Hint: Start by considering matrices like $B = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$.

Solution:

Solution (continued):

Exercise 3 (5 points): Let V be a real vector space. Suppose that v_1, v_2, v_3, v_4 are vectors in V which are linearly independent. Show that the vectors

$$v_1, \quad v_1 + v_2, \quad v_1 + v_2 + v_3, \quad v_1 + v_2 + v_3 + v_4$$

are also linearly independent.

Solution:

Solution (continued):

Exercise 4 (5 points): Let a, b be real numbers. Consider the following matrix:

$$A = \begin{pmatrix} 1 & a & b & 0 \\ 0 & 1 & a & b \\ 0 & 0 & 1 & a \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

For which values a and b is A invertible? For these values, write down A^{-1} in terms of a and b (simplify all expressions).

Solution:

Solution (continued):

Exercise 5 (8 points): Which of the following are subspaces of $\mathbb{R}^2$? If you think the given set is a subspace, prove it. If you think the given set is not a subspace, show that it isn't.

(1) The set V of vectors $\left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right]$ such that $|x_1| = |x_2|$.
(2) The set V of vectors $\left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right]$ such that $x_1 - 2x_2 = 0$.

Which of the following maps are linear transformations? (Justify your answer)

(3) $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(\left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right]) = \left[\begin{smallmatrix} x_1 \cdot x_2 \\ x_1 + x_2 \end{smallmatrix}\right]$.
(4) $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(\left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right]) = \left[\begin{smallmatrix} x_1 + x_2 \\ x_1 - x_2 \end{smallmatrix}\right]$.

Solution:

Solution (continued):

Exericse 6 (8 points): Consider the following matrix:

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \end{bmatrix}$$

(1) Find a basis for $\mathrm{Ker}(A)$. What is $\dim(\mathrm{Ker}(A))$?
(2) Find a basis for $\mathrm{Im}(A)$. What is $\dim(\mathrm{Im}(A))$?

Solution:

Solution (continued):

Exercise 7 (8 points): Let $V \subseteq P_2(\mathbb{R})$ be the set of polynomials $f(X) = a_0 + a_1 X + a_2 X^2$ such that $f(1) = 0$.

(1) Show that V is a subspace of $P_2(\mathbb{R})$.
(2) Find a basis of V. What is $\dim(V)$?

Solution:

Solution (continued):

Exercise 8 (8 points): Consider the vectors

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \qquad v_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix}, \qquad v_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \end{bmatrix}$$

(1) Show that v_1, v_2, v_3 are linearly independent in $\mathbb{R}^4$.
(2) Construct an orthonormal basis of $V = \mathrm{Span}(v_1, v_2, v_3)$.

Solution:

Solution (continued):

Exercise 9 (8 points): Find all least squares solutions of the following system of equations:

$$\begin{aligned} X + Y + 2Z &= 2 \\ 2X + 2Y + 3Z &= 1 \\ 3X + 3Y + 4Z &= 3 \end{aligned}$$

Solution:

Solution (continued):

Exercise 10 (10 points): Let $\mathcal{C}([-2,2])$ denote the vector space of continuous functions $f : [-2,2] \to \mathbb{R}$, equipped with the inner product:

$$\langle f, g \rangle = \int_{-2}^{2} f(t)g(t)dt$$

(1) Consider the function $f(t) = t$. Compute $||f||$.
(2) Construct an orthonormal basis (with respect to the inner product $\langle -, - \rangle$ above) of the sub-space $P_1(\mathbb{R})$ of polynomials of degree ≤ 1.
(3) Consider the function $g(t) = t^3$. Compute $\text{Proj}_{P_1(\mathbb{R})}(g)$, and draw it (together with $g(t)$) on the interval $[-2,2]$.

Solution:

Solution (continued):

Exercise 11 (8 points): Compute the determinant of the following matrix (show your work):

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 2 & 2 & 2 \\ 1 & 0 & 3 & 0 & 0 \\ 1 & 0 & 3 & 4 & 4 \\ 1 & 0 & 3 & 0 & 5 \end{pmatrix}$$

Solution:

Solution (continued):

Exercise 12 (12 points): Consider the following matrix:

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

(1) Write down the characteristic polynomial $f_A(X)$. What are the real eigenvalues of A, and their corresponding algebraic multiplicities?
(2) A is diagonalizable. Find a basis of $\mathbb{R}^4$ consisting of eigenvectors of A.
(3) Find an orthonormal basis of $\mathbb{R}^4$ consisting of eigenvectors of A.
(4) Use your answer to compute A^7 by diagonalizing A.

Solution:

Solution (continued):