ORTHOGONALITY

1. Dot Product

Recall, the *dot product* of two vectors $\vec{v}, \vec{w} \in \mathbb{R}^n$ is defined to be

$$\vec{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}, \vec{w} = \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix}, \vec{v} \cdot \vec{w} = v_1 w_1 + \ldots + v_n w_n$$

The length of a vector, $||\vec{v}||$, is defined by

$$||\vec{v}|| = \sqrt{\vec{v} \cdot \vec{v}}.$$

Notice, $||\vec{v}|| = 0 \iff \vec{v} = \vec{0}$. A vector $\vec{u} \in \mathbb{R}^n$ is a *unit vector* if $||\vec{u}|| = 1$. It follows from basic properties of the dot product, that if $\vec{v} \neq 0$, then $\vec{u} = \frac{1}{||\vec{v}||}\vec{v}$ is a unit vector. Indeed,

$$\vec{u} \cdot \vec{u} = \left(\frac{1}{||\vec{v}||} \vec{v} \right) \cdot \left(\frac{1}{||\vec{v}||} \vec{v} \right) = \frac{1}{||\vec{v}||^2} \vec{v} \cdot \vec{v} = \frac{||\vec{v}||^2}{||\vec{v}||^2} = 1.$$

2. Orthogonality

Two vectors $\vec{v}$ and $\vec{w}$ are said to be *perpendicular* or *orthogonal* if

$$\vec{v} \cdot \vec{w} = 0.$$

Geometrically, means that if the vectors non-zero, then they meet at $90°$. If V is a subspace of $\mathbb{R}^n$, then $\vec{w}$ is *orthogonal* to V if

$$\vec{w} \cdot \vec{v} = 0 \text{ for all } \vec{v} \in V.$$

Observe $\vec{w}$ is orthogonal to $V \iff \vec{w}$ is orthogonal to each $\vec{v}_1, \ldots, \vec{v}_m$ where these vectors form a basis of V. I leave the details of this verification as an exercise.

Vectors $\vec{u}_1, \ldots, \vec{u}_m$ are said to be *orthonormal* if

(1) They are unit, i.e. $\vec{u}_i \cdot \vec{u}_i = 1$ for $i = 1, \ldots, m$
(2) They are pairwise orthogonal, i.e. $\vec{u}_i \cdot \vec{u}_j = 0$, $i \neq j$.

EX:

$$\vec{u}_1 = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} \text{ and } \vec{u}_2 = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$$

are orthonormal for any θ.

Theorem 2.1. *Orthonormal vectors have the following properties:*

(1) *If $\vec{u}_1, \ldots, \vec{u}_m$ are orthonormal, then they are linearly independent*
(2) *If $V \subset \mathbb{R}^n$ is a subspace with $\dim(V) = m$ and $\vec{u}_1, \ldots, \vec{u}_m \in V$ are orthonormal, then $\vec{v}_1, \ldots, \vec{v}_m$ are a basis of V.*
(3) *If $V \subset \mathbb{R}^n$ is a subspace with $\dim(V) = m$, then there are $\vec{u}_1, \ldots, \vec{u}_m \in V$ which are orthonormal (and hence form an* orthonormal basis *of V).*

Proof. Proof of (1): Suppose $c_1\vec{u}_1 + \ldots + c_m\vec{u}_m = \vec{0}$. This means

$$(c_1\vec{u}_1 + \ldots + c_m\vec{u}_m) \cdot \vec{u}_i = \vec{0} \cdot \vec{u}_i.$$

Hence, $c_1\vec{u}_1 \cdot \vec{u}_i + \cdots + c_i\vec{u}_i \cdot \vec{u}_i + \cdots + c_m\vec{u}_m \cdot \vec{u}_i = 0$ Tnis means, $c_i = 0$. As i was arbitrary, the only relation is trivial one. Claim (2) follows immediately from this and earlier work on dimension. Finally, for the proof of claim (3), see footnote on pg. 205 of the text – you can also use the Gram-Schmidt algorithm. □

3. Orthogonal Projection

We have the following generalization of concept of orthogonal projection in $\mathbb{R}^2$:

Theorem 3.1. *Fix a subspace $V \subset \mathbb{R}^n$. For any $\vec{x} \in \mathbb{R}^n$ we can write*

$$\vec{x} = \vec{x}^{||} + \vec{x}^{\perp}$$

where $\vec{x}^{||} \in V$ and $\vec{x}^{\perp}$ is orthogonal to V. This decomposition is unique. Furthermore, the transformation

$$\mathrm{proj}_V : \mathbb{R}^n \to \mathbb{R}^n$$

defined by $\mathrm{proj}_V(\vec{x}) = \vec{x}^{||}$ is a linear transformation.

The map proj_V is called *orthogonal projection of $\vec{x}$ onto V.*

Proof. Pick an orthonormal basis $\vec{u}_1, \ldots, \vec{u}_m$ of V. Let

$$\vec{y} = (\vec{x} \cdot \vec{u}_1)\vec{u}_1 + \ldots + (\vec{x} \cdot \vec{u}_m)\vec{u}_m$$

Observe, $\vec{y} \in V$ and $\vec{y} \cdot \vec{u}_i = \vec{x} \cdot \vec{u}_i$. Let $\vec{z} = \vec{x} - \vec{y}$. So

$$\vec{z} \cdot \vec{u}_i = (\vec{x} - \vec{y}) \cdot \vec{u}_i = \vec{x} \cdot \vec{u}_i - \vec{y} \cdot \vec{u}_i = 0.$$

Hence, $\vec{z}$ is orthogonal to a basis of V and hence is orthogonal to V itself. That is, we may write

$$\vec{x}^{||} = \vec{y} = (\vec{x} \cdot \vec{u}_1)\vec{u}_1 + \ldots + (\vec{x} \cdot \vec{u}_m)\vec{u}_m$$

and

$$\vec{x}^{\perp} = \vec{z} = \vec{x} - \vec{x}^{||}$$

and obtain desired decomposition. The uniqueness and linearity of proj_V is left as an exercise for the reader. □

A consequence of the above proof is the following useful formula for proj_V in terms of an orthonormal basis of V:

Theorem 3.2. *If $V \subset \mathbb{R}^n$ a subspace and $\vec{u}_1, \ldots, \vec{u}_m$ an orthonormal basis of V, then*

$$proj_V(\vec{x}) = (\vec{x} \cdot \vec{u}_1)\vec{u}_1 + \ldots + (\vec{x} \cdot \vec{u}_m)\vec{u}_m.$$

Orthogonal Complement Given a subspace $V \subset \mathbb{R}^n$. The *orthogonal complement*, $V^{\perp} \subset \mathbb{R}^n$ of V is defined by

$$V^{\perp} = \{\vec{x} \in \mathbb{R}^n : \vec{x} \cdot \vec{v} = 0 \text{ for all } \vec{v} \in V\}.$$

EXAMPLE: In $\mathbb{R}^3$, $(\mathrm{span}(\vec{e}_1))^{\perp} = \mathrm{span}(\vec{e}_2, \vec{e}_3)$.

Theorem 3.3. *Properties of orthogonal complement for $V \subset \mathbb{R}^n$:*

1. $V^{\perp} = \ker(\mathrm{proj}_V)$ – *in particular, $V^{\perp}$ is a subspace of $\mathbb{R}^n$*
2. $V \cap V^{\perp} = \left\{\vec{0}\right\}$
3. $\dim(V) + \dim(V^{\perp}) = n$

(4) $(V^\perp)^\perp = V$.

Proof. Claim (1): Follows from the definition. Claim (2): If $\vec{x} \in V$ and $\vec{x} \in V^\perp$, then $\vec{x}$ is perpendicular to itself. That is $||\vec{x}||^2 = \vec{x}\cdot\vec{x} = 0 \Rightarrow \vec{x} = 0$. Claim (3): We have $V = \mathrm{Im}\,(\mathrm{proj}_V)$ and $V^\perp = \ker(\mathrm{proj}_V)$. Hence, the rank-nullity theorem implies

$$n = \dim(\mathrm{Im}\,(\mathrm{proj}_V)) + \dim(\ker(\mathrm{proj}_V)) = \dim(V) + \dim(V^\perp).$$

Claim (4): Clear $V \subset (V^\perp)^\perp$. By Claim (3):

$$n = \dim(V) + \dim(V^\perp) = \dim(V^\perp) + \dim((V^\perp)^\perp) \Rightarrow \dim(V) = \dim((V^\perp)^\perp)$$

Hence, $V = (V^\perp)^\perp$. $\square$

EXAMPLE: Let $V = \mathrm{span}\left(\begin{bmatrix}1\\2\\1\end{bmatrix}, \begin{bmatrix}0\\-1\\2\end{bmatrix}\right)$. Determine $V^\perp$ Observe,

$$\vec{x} = \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} \in V^\perp \iff \vec{x}\cdot\begin{bmatrix}1\\2\\1\end{bmatrix} = \vec{x}\cdot\begin{bmatrix}0\\-1\\2\end{bmatrix} = 0$$

That is

$$x_1 + 2x_2 + x_3 = 0 = -x_2 + 2x_3$$

I.e.,

$$\begin{bmatrix}1 & 2 & 1\\0 & -1 & 2\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

Hence,

$$V^\perp = \ker\begin{bmatrix}1 & 2 & 1\\0 & -1 & 2\end{bmatrix}$$

$$\mathrm{rref}\begin{bmatrix}1 & 2 & 1\\0 & -1 & 2\end{bmatrix} = \begin{bmatrix}1 & 0 & 5\\0 & 1 & -2\end{bmatrix}$$

so

$$V^\perp = \mathrm{span}\left(\begin{bmatrix}-5\\2\\1\end{bmatrix}\right).$$