Mathematics 201, Spring 2017: Assignment #10

Due: **In lecture, Friday, May 5th**

Instructions: Please ensure your name, your TA's name and your section number appear on the first page. Also that your answers are legible and all pages are stapled. Page numbers refer to the course text.

Problem #1. Show that if $A \in \mathbb{R}^{n\times n}$ has $\text{rank}(A) = 1$, then any $\vec{v} \in \text{im}(A), \vec{v} \neq \vec{0}$, is an eigenvector of A.

Problem #2. Suppose that $A \in \mathbb{R}^{n\times n}$ satisfies $A^2 - 4A = -4I_2$. If $\vec{v}$ is an eigenvector of A, determine the possible values for the associated eigenvalue.

Problem #3. Let V be the subspace given by the plane $x_1 + x_2 - x_3 = 0$ in $\mathbb{R}^3$. Let P be the matrix given by orthogonal projection onto V.

a) Determine the eigenvalues of P and their algebraic multiplicity (Hint: Reason geometrically).

b) Diagonalize P.

Problem #4. Let $Q \in \mathbb{R}^{3\times 3}$ be an orthogonal matrix.

a) Show that if $\vec{v} \in \mathbb{R}^3$ is an eigenvector of Q and $\vec{w} \in \mathbb{R}^3$ is perpendicular to $\vec{v}$, then $Q\vec{w}$ is also perpendicular to $\vec{v}$. (Hint: As Q is orthogonal, $Q\vec{v}$ is orthogonal to $Q\vec{w}$.)

b) Show that there is a line $L \subset \mathbb{R}^3$ so that L and its orthogonal complement, the plane $L^\perp$, are invariant sets of Q. (A subspace, V, is an invariant set of Q, if $\vec{v} \in V \Rightarrow Q\vec{v} \in V$).

Problem #5. Let $Q = \frac{1}{3}\begin{bmatrix} 2 & -1 & -2 \\ 2 & 2 & 1 \\ 1 & -2 & 2 \end{bmatrix}$.

a) Verify that Q is orthogonal.

b) Determine the invariant line L and invariant plane $L^\perp$ whose existence is ensured by the previous problem. (Hint: Use that only (real) eigenvalues of an orthogonal matrix are ± 1).

c) Describe geometrically what multiplication by Q does. (Hint: Consider what Q does to an orthonormal basis of $L^\perp$.)

Problem #6. Consider the matrix $A = \begin{bmatrix} 1 & k+1 & 0 \\ 0 & 1 & k^2-1 \\ 0 & 0 & 1 \end{bmatrix}$ where $k \in \mathbb{R}$.

a) Verify that the only eigenvalue of A is 1 with algebraic multiplicity 3.

b) Determine how the geometric multiplicity of the eigenvalue 1 changes as you vary k.

Problem #7. For the following matrices, explain why they cannot be diagonalized.

a) $\begin{bmatrix} 3 & 0 & 4 \\ 0 & 5 & 0 \\ -4 & 0 & 3 \end{bmatrix}$.

b) $\begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix}$.

Problem #8. Determine whether the following linear transformations are diagonalizable.

a) $T : P_2 \to P_2$ given by $T(p)(x) = p(x-3)$.

b) $T : P_2 \to P_2$ given by $T(p)(x) = xp'(x)$.

Problem #9. For the following matrices determine A^{11}.

a) $A = \begin{bmatrix} -2 & -2 \\ 3 & 3 \end{bmatrix}$

b) $A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix}$.

Problem #10. Find an orthonormal eigenbasis of the following matrices:

a) $\begin{bmatrix} 2 & 2 \\ 2 & 1 \end{bmatrix}$.

b) $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ 1 & -1 & 1 \end{bmatrix}$.

Book Problems.

a) Section 7.1 #38, #62
b) Section 7.2 #22, #28, #48
c) Section 7.3 #14, #40
d) Section 7.4 #54
e) Section 8.1 #10, #22