Linear Algebra Problem Set # 10
Selected Solutions

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Problem #1.

Suppose $A \in \mathbb{R}^{n \times n}$ has rank(A)=1, and let $\vec{v}$ $\in$im(A), $\vec{v} \neq \vec{0}$. We need to show that $\vec{v}$ is an eigenvector of A. Since A has rank=1, we know that $\dim(\text{im}(A)) = 1$, so let $\vec{a}$ denote the single element of the basis of im(A). Since $\vec{v}$ lies in the image of A, we have that $\vec{v} = k\vec{a}$ for some $k \in \mathbb{R}$. So now we plug and chug:

$$A\vec{v} = A(k\vec{a}) = k(A\vec{a}) = km\vec{a} = m\vec{v}$$

for $m \in \mathbb{R}$. So $\vec{v}$ is an eigenvector of A, as we wished to show. ♠

Problem #2.

Suppose that $a \in \mathbb{R}^{n \times n}$ satisfies the relation $A^2 - 4A = -4I_n$. We want to find what the possible eigenvalues of A are, which is to say that we want to find all values of λ for which there is one (or perhaps more) associated $\vec{v} \in \mathbb{R}^n$ such that the following holds:

$$(A - \lambda I_n)\vec{v} = \vec{0}$$

Note that we can do the following::

$$A^2 - 4A = -4I_n \Rightarrow A^2 - 4A + 4I_n = 0 \Rightarrow (A - 2I_n)(A - 2I_n) = 0$$

(Multiply this out to convince yourself of this.) From this, it follows that:

$$(A - 2I_n)\vec{v} = \vec{0}$$

Which suggests that the possible eigenvalues of the matrix are $\lambda = 2$. ♠

Problem #3.

(a) The following is true of V:

$$V = \text{span}\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right)$$

This is true since any vector $\vec{v}$ lying along V must obey the equation $v_1 + v_2 - v_3 = 0 \Rightarrow v_1 + v_2 = v_3$, and hence has the form $(v_1, v_2, v_1 + v_2)$. Decomposing this gives the two basis vectors $\vec{v}_1$ and $\vec{v}_2$ above.

Furthermore, we have that the normal vector to V is $\vec{v}_3 = (1, 1, -1)$, which follows directly from the equation of the plane. Taking $\vec{x} = (x_1, x_2, x_3) \in V$ we have:

$$(1, 1, -1) \cdot (x_1, x_2, x_3) = x_1 + x_2 - x_3$$

which we know from the equation of the plane is equal to 0. Note that $\text{proj}_V(\vec{v}_1) = \vec{v}_1$ and $\text{proj}_V(\vec{v}_2) = \vec{v}_2$. So the eigenvalues are $\lambda = 0, 1$, with almu(0)=1 and almu(1)=2. The eigenvectors are the basis vectors of the space (with eigenvector 1) and the normal vector (with eigenvector 0). ♠

(b) The diagonalization of $\text{proj}_V : \mathbb{R}^3 \to \mathbb{R}^3$ is given by:

$$B = S^{-1}[\text{proj}_V]_{B_V} S = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

where

$$S = \left[\vec{v}_1 \mid \vec{v}_2 \mid \vec{v}_3 \right]$$

♠

Problem #4.

(a) Suppose that $\vec{v}$ is an eigenvector of $Q \in \mathbb{R}^{n \times n}$, for Q an orthogonal matrix, and suppose that $\vec{w}$ is orthogonal to $\vec{v}$. Since Q is orthogonal, we have that

$$0 = |\vec{v} \cdot \vec{w}| = |Q\vec{v} \cdot Q\vec{w}| = |k\vec{v} \cdot Q\vec{w}| = |k||\vec{v} \cdot Q\vec{w}| \Rightarrow 0 = |v \cdot Q\vec{w}|$$

Hence $\vec{v}$ is orthogonal to $Q\vec{w}$. ♠

(b) By (a), if $\vec{v}$ is an eigenvector of an orthogonal matrix Q, then if $\vec{w}$ is orthogonal to $\vec{v}$, so is $Q\vec{w}$. So we let $L = \text{span}(\vec{v})$. This is clearly an invariant set of Q since, for any $\vec{x} \in L$, we have $Q\vec{x} = Q(k\vec{v}) = kQ\vec{v} = km\vec{v} = m\vec{x}$.

Correspondingly, let $\vec{w}, \vec{u} \in \mathbb{R}^3$ be orthogonal to $\vec{v}$ (with $\vec{w}$ and $\vec{u}$ linearly independent). Let $L^\perp = \text{span}(\vec{w}, \vec{u})$. Consider some $\vec{y} \in L^\perp$, with $\vec{y} = a\vec{w} + b\vec{u}$. Then

$$|\vec{v} \cdot Q\vec{x}| = |\vec{v} \cdot Q(a\vec{w} + b\vec{u})| = |\vec{v} \cdot (Qa\vec{w}) + \vec{v} \cdot (Qb\vec{u})| =$$

$$|a(\vec{v} \cdot (Q\vec{w})) + b(\vec{v} \cdot (Q\vec{u}))| = |a(0) + b(0)| = 0$$

And so $Q\vec{x} \in L^\perp$, which shows that $L^\perp$ is an invariant set of Q. ♠

Problem #5.

(a) To verify Q is orthogonal, we check whether $Q^T Q = I_3$:

$$\frac{1}{3}\begin{bmatrix} 2 & -1 & -2 \\ 2 & 2 & 1 \\ 1 & -2 & 2 \end{bmatrix} \cdot \frac{1}{3}\begin{bmatrix} 2 & 2 & 1 \\ -1 & 2 & -2 \\ -2 & 1 & 2 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

So Q is indeed an orthogonal matrix. ♠

(b) We know that the only real eigenvalues of an orthogonal matrix are $\lambda = 1$ and $\lambda = -1$. So we need to find a vector that Q scales by 1 or -1. By inspection we can see that $(1, 1, -1)$ is such a vector with $\lambda = 1$, since $\frac{1}{3}(2 \times 1 + -1 \times 1 + -2 \times -1) = 1$, $\frac{1}{3}(2 \times 1 + 2 \times 1 + 1 \times -1) = 1$, and $\frac{1}{3}(1 \times 1 + -2 \times 1 + 2 \times -1) = -1$

Now, from Problem #4, we can find L fairly easily by just finding an eigenvector of Q:

$$L = \text{span}\left(\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \right)$$

To find $L^\perp$, we simply need to find two vectors that are perpendicular to $(1, 1, -1)$. Note that we can just find vectors which span a plane to which $(1, 1, -1)$ is a normal vector. Recalling Problem #3, we get that such a plane is in fact

$$L^\perp = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 + x_2 - x_3 = 0\}$$

It's easy to see this: take some vector lying on that plane, (x_1, x_2, x_3), and take its dot product with $(1, 1, -1)$: $(x_1, x_2, x_3) \cdot (1, 1, -1) = x_1 + x_2 - x_3 = 0$. So we get that:

$$L^\perp = \text{span}\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right)$$

♠

(c) If we run Gram-Schmidt on the basis of $L^\perp$ we get the following orthonormal basis of $L^\perp$:

$$\mathfrak{B} = \{(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}), (\frac{-1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}})\}$$

(Check for yourself that these still lie on the original plane.) Now, we apply Q to these:

$$\frac{1}{3}\begin{bmatrix} 2 & -1 & -2 \\ 2 & 2 & 1 \\ 1 & -2 & 2 \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$\frac{1}{3}\begin{bmatrix} 2 & -1 & -2 \\ 2 & 2 & 1 \\ 1 & -2 & 2 \end{bmatrix} \cdot \begin{bmatrix} \frac{-1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{bmatrix} = \begin{bmatrix} \frac{-2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ \frac{-1}{\sqrt{6}} \end{bmatrix}$$

This suggests a rotation. Recalling the formula $\vec{x} \cdot \vec{y} = \|\vec{x}\| \|\vec{y}\| \cos\theta$, we take the dot product of the original vectors and the resultant vectors and find the inverse cosine:

$$\|\vec{u_1}\|^2 \cos\theta = \vec{u_1} \cdot Q\vec{u_1} = \frac{1}{2} \Rightarrow \cos\theta = \frac{1}{2} \Rightarrow \theta = \arccos\frac{1}{2} = \frac{\pi}{3}$$

$$\|\vec{u_2}\|^2 \cos\theta = \vec{u_2} \cdot Q\vec{u_2} = \frac{1}{2} \Rightarrow \cos\theta = \frac{1}{2} \Rightarrow \theta = \arccos\frac{1}{2} = \frac{\pi}{3}$$

So we conclude that Q represents a rotation through $\frac{\pi}{3}$.

Problem #6.

(a) We find the secular equation of A by taking $\det(A - \lambda I_3)$:

$$f_A(\lambda) = \begin{vmatrix} 1-\lambda & k+1 & 0 \\ 0 & 1-\lambda & k^2-a \\ 0 & & 1-\lambda \end{vmatrix} = (1-\lambda)\begin{vmatrix} 1-\lambda & k^2-1 \\ 0 & 1-\lambda \end{vmatrix} = (1-\lambda)^3$$

Since λ is a root of $f_A(\lambda)$ with multiplicity 3 $\lambda = 1$ has algebraic multiplicity 3. ♠.

(b) To determine this we examine rank($A - I_3$):

$$A - I_3 = \begin{bmatrix} 0 & k+1 & 0 \\ 0 & 0 & k^2-1 \\ 0 & 0 & 0 \end{bmatrix}$$

Note that if $k = -1$, rank($A - I_3$)=0, in which case gemu(λ)=3, and if $k = 1$, rank($A - I_3$)=1, in which case gemu(λ)=2. In all other cases where $k \in \mathbb{R}$, rank($A - I_3$)=2, and so gemu(λ)=1. ♠

Problem #7.

(a) Note that $f_A(\lambda) = -(\lambda - 5)(\lambda^2 - 6\lambda + 25)$, which has roots $\lambda = 5, 3 + 4i, 3 - 4i$. Consequently A cannot be similar to any diagonal matrix with real eigenvalues, and hence is not diagonalizable over $\mathbb{R}$ (though it may be diagonalizable over $\mathbb{C}$). ♠

(b) Let's look at the secular equation again: $f_A(\lambda) = (2 - \lambda)^2$. In this case, we have purely real roots: $\lambda = 2$, almu(λ)=2. But note that gemu(2)=2−rank($A - 2I_2$)=$2 - 1$=1. Since the geometric multiplicity of an eigenvalue is equal to the dimension of its corresponding eigenspace, we have that the only eigenspace of A is of dimension 1. Consequently, the basis consisting of all eigenvectors of A will have only a single element, which means that it cannot form an eigenbasis (i.e., it cannot span $\mathbb{R}^2$). Therefore, A is not diagonalizable.♠

Problem #8.

(a) Note that T is diagonalizable iff we can find an eigenbasis of T. So we need to find the eigenvectors of T and ask whether they span P_2. Let $p(x) = a_0 + a_1 x + a_2 x^2$. Now we apply T:

$$T(p(x)) = a_0 + a_1(x - 3) + a_2(x - 3)^2 = a_0 + a_1 x - 3a_1 + a_2 x^2 - 6a_2 x + 9a_2$$

$$= (a_0 - 3a_1 + 9a_3) + (a_1 - 6a_2)x + a_2 x^2$$

So the corresponding matrix is:

$$[T]_B = \begin{bmatrix} 1 & -3 & 9 \\ 0 & 1 & -6 \\ 0 & 0 & 1 \end{bmatrix}$$

with secular equation $f_T(\lambda) = (1 - \lambda)^3$. So $[T]_B$ has eigenvalue $\lambda = 1$ with almu(λ)=3. Note what happens when we find ker($A - I_3$):

$$\left[\begin{array}{ccc|c} 0 & -3 & 9 & 0 \\ 0 & 0 & -6 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

This gives us that $[T]_B$ has eigenvector $\vec{v} = (1, 0, 0)$, which means that T has eigenvector $p(x) = 1$. Clearly, span($p(x) = 1$)$\neq P_2$, so T is not diagonalizable. ♠

(b) Let $p(x) = a_0 + a_1 x + a_2 x^2$. Then $T(p(x)) = xp'(x) = x(a_1 + 2a_2 x) = a_1 x + 2a_2 x^2$. If $p(x)$ is an eigenvalue of T, then we have that $T(p(x)) = \lambda p(x)$, which implies in this case that $a_0 = 0$, $\lambda a_1 = a_1$, and $\lambda a_2 = 2a_2$. So the eigenvectors of T are $p_1(x) = 1$, with eigenvalue $\lambda = 0$, $p_2(x) = x$, with eigenvalue $\lambda = 1$, and $p_3(x) = x^2$, with eigenvalue $\lambda = 2$.

These clearly do span P_2, since for arbitrary $q(x)$, we can write it in some superposition of p_1, p_2, and p_3. Consequently, T is diagonalizable.

Another way to see this is, again, to look at $[T]_B$:

$$[T]_B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

This matrix has eigenvalues $\lambda = 0, 1, 2$, and the corresponding eigenvectors are just the standard basis vectors of $\mathbb{R}^3$. Thus since we have an eigenbasis of $\mathbb{R}^3$, and $\mathbb{R}^2 \cong P_2$, we also have an eigenbasis of P_2 under the usual basis transformation. ♠

Problem #9.

(a) We diagonalize! The secular equation of A is $f_A(\lambda) = (-2-\lambda)(3-\lambda) + 6 = \lambda(\lambda - 1)$, so $\lambda = 0, 1$, with associated eigenvectors $\vec{v}_1 = (-1, 1)$, $\vec{v}_2 = (-2, 3)$. So now, we compute:

$$A^{11} = \begin{bmatrix} -1 & -2 \\ 1 & 3 \end{bmatrix} \cdot \begin{bmatrix} 0^{11} & 0 \\ 0 & 1^{11} \end{bmatrix} \cdot \begin{bmatrix} -3 & -2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & -2 \\ 1 & 3 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} -2 & -2 \\ 3 & 3 \end{bmatrix} = A$$

♠

(b) We do the same thing. Since A is upper-triangular, the eigenvalues are $\lambda = -1, 1, 0$, with corresponding eigenvectors $\vec{v}_1 = (-1, -2, 2)$, $\vec{v}_2 = (1, 0, 0)$, and $\vec{v}_3 = (-1, 1, 0)$. So now we compute!

$$A^{11} = \begin{bmatrix} -1 & 1 & -1 \\ -2 & 0 & 1 \\ 2 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} (-1)^{11} & 0 & 0 \\ 0 & 1^{11} & 0 \\ 0 & 0 & 0^{11} \end{bmatrix} \cdot \frac{1}{2} \begin{bmatrix} 0 & 0 & 1 \\ 2 & 2 & 3 \\ 0 & 2 & 2 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} -1 & 1 & -1 \\ -2 & 0 & 1 \\ 2 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & -1 \\ 2 & 2 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 2 & 2 & 4 \\ 0 & 0 & 2 \\ 0 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix} = A$$

♠

Problem #10.

(a) We know that if A is symmetric (which it is), then it is orthogonally diagonalizable. Hence, all we need to do to find an orthonormal eigenbasis is find the eigenvectors and normalize them. The characteristic polynomial of A is $f_A(\lambda) = (2-\lambda)(1-\lambda) - 4 = \lambda^2 - 3\lambda - 2$, with roots $\lambda = \frac{1}{2}(3+\sqrt{17}), \frac{1}{2}(3-\sqrt{17})$. The associated eigenvectors are $\vec{v}_1 = (\frac{1}{4}(1+\sqrt{17}), 1), \vec{v}_1 = (\frac{1}{4}(1-\sqrt{17}), 1)$. Hence our eigenbasis is:

$$\mathfrak{B} = \{\frac{1}{\|\vec{v}_1\|}(\frac{1}{4}(1+\sqrt{17}), 1), \frac{1}{\|\vec{v}_2\|}(\frac{1}{4}(1-\sqrt{17}), 1)\}$$

where

$$\|\vec{v}_1\| = \frac{1}{2}\sqrt{\frac{1}{2}(17+\sqrt{17})}$$

$$\|\vec{v}_2\| = \frac{1}{2}\sqrt{\frac{1}{2}(17-\sqrt{17})}$$

♠

(b) The same thing holds here as in (a). The secular equation of A is $f_A(\lambda) = -\lambda(\lambda - 2)(\lambda+1)$, so the eigenvalues are $\lambda = 2, -1, 0$, with corresponding eigenvectors $\vec{v}_1 = (1, -1, 2), \vec{v}_2 = (-1, 1, 1), \vec{v}_3 = (1, 1, 0)$. So our orthonormal eigenbasis is:

$$\mathfrak{B} = \{(\frac{1}{\sqrt{6}}, \frac{-1}{\sqrt{6}}, \frac{2}{\sqrt{6}}), (\frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}), (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0)\}$$

♠