※1.

$$I_n = -(A^3+A^2-2A) = -A^3-A^2+2A = A(-A^2-A+2I_n)$$

Furtheremore, $I_n = (-A^2-A+2I_n)A$.

Hence, A is invertible and $A^{-1} = -A^2-A+2I_n$.

※2.

a) Let $A = \begin{bmatrix} | & | & | \\ \vec{v_1} & \vec{v_2} & \vec{v_3} \\ | & | & | \end{bmatrix}$. Then

$$AF_{1,2} = A\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} A\begin{bmatrix}0\\1\\0\end{bmatrix} & A\begin{bmatrix}1\\0\\0\end{bmatrix} & A\begin{bmatrix}0\\0\\1\end{bmatrix} \end{bmatrix} = \begin{bmatrix} | & | & | \\ \vec{v_2} & \vec{v_1} & \vec{v_3} \\ | & | & | \end{bmatrix}.$$

$\therefore$ $AF_{1,2}$ is obtained from A by swapping the first and second columns of A.

b) $\mathrm{Im}(A) = \mathrm{Span}\{\vec{v_1}, \vec{v_2}, \vec{v_3}\} = \mathrm{Im}(AF_{1,2})$.

c) Not necessarily true.

Consider $\begin{cases} A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \rightarrow \ker(A) = \mathrm{Span}\left\{\begin{bmatrix}1\\0\\-1\end{bmatrix}\right\} \\ AF_{1,2} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \rightarrow \ker(AF_{1,2}) = \mathrm{Span}\left\{\begin{bmatrix}0\\1\\-1\end{bmatrix}\right\}. \end{cases}$

※3.

a) $\begin{bmatrix} 2 & -2 & 0 & 4 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 3 & 1 \end{bmatrix} \xrightarrow[\text{Elimination}]{\text{Gauss}} \begin{bmatrix} 1 & 0 & 0 & 5 & 1 \\ 0 & 1 & 0 & 3 & 1 \\ 0 & 0 & 1 & -8 & -1 \end{bmatrix}$ $\therefore$ rank = 3, nullity = 2.

b) $\begin{bmatrix} 0 & 1 & 2 & 0 \\ 1 & 2 & 0 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix} \xrightarrow[\text{Elimination}]{\text{Gauss}} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ $\therefore$ rank = 3, nullity = 1.

※4.

Claim: $\mathrm{Im}(A) \subseteq \mathrm{Im}(B)$.

Suppose $\vec{v} \in \mathrm{Im}(A)$.

$\Rightarrow \vec{v} = A\vec{x}$ for some $\vec{x}$ $(\in \mathbb{R}^5)$

$\Rightarrow \vec{v} = BC\vec{x} = B(C\vec{x})$

$\Rightarrow \vec{v} \in \mathrm{Im}(B)$. //

Hence, $\mathrm{rank}(A) = \dim(\mathrm{Im}(A)) \leq \dim(\mathrm{Im}(B)) = \mathrm{rank}(B) \leq 3$

$\because B$ is a 5×3 matrix.

Since $\mathrm{rank}(A) \leq 3$, A cannot be invertible.

※5.

Note that all $A\vec{v}$, $A^2\vec{v}$, and $A^3\vec{v}$ are in $\mathrm{Im}(A)$.

Since $\dim(\mathrm{Im}(A)) = \mathrm{rank}(A) \leq 2$, three vectors $A\vec{v}, A^2\vec{v}, A^3\vec{v}$ in $\mathrm{Im}(A)$ have to be linearly dependent. Hence, $\vec{v}, A\vec{v}, A^2\vec{v}, A^3\vec{v}$ cannot be linearly independent.

※6.

a) Claim: $\ker(A) \supseteq \mathrm{Im}(A)$.

Suppose $\vec{v} \in \mathrm{Im}(A)$

$\Rightarrow \vec{v} = A\vec{x}$ for some $\vec{x}$ $(\in \mathbb{R}^4)$.

$\Rightarrow \vec{0} = 0\cdot\vec{x} = A^2\vec{x} = A(A\vec{x}) = A\vec{v}$

$\Rightarrow \vec{v} \in \ker(A)$.

$\therefore$ $\mathrm{Im}(A)$ is a subspace of $\ker(A)$.

b) By the rank-nullity theorem, $\dim(\ker(A)) + \dim(\mathrm{Im}(A)) = 4$.

Since $\dim(\ker(A)) \geq \dim(\mathrm{Im}(A))$ by a), it follows that $\mathrm{rank}(A) = \dim(\mathrm{Im}(A)) \leq 2$.

※ 7.

a) <u>For some T.</u>

Ex. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$ rank([T]) = null([T]) = 2.

Counter-Ex. $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ rank([T]) = 0 < 4 = null([T]).

b) <u>For some T.</u>

Ex. $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$ rank([T]) = 2, $T(\vec{e}_1) = \vec{e}_2$.

Counter-Ex. $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ rank([T]) = 0.

c) <u>For no T.</u>

Since $T(\vec{e}_1) = \vec{e}_1$ & $T(\vec{e}_2) = \vec{e}_1 + \vec{e}_2$, $Im(T) \supseteq Span\{\vec{e}_1, \vec{e}_1 + \vec{e}_2\}$,

and therefore, rank(T) = dim(Im(T)) ≥ 2.

Then, rank(T) + null(T) ≥ 2 + 3 = 5. This contradicts the rank-nullity thm.

d) <u>For no T.</u>

By rank-nullity thm, rank(T) = 4 − null(T) = 4 − 1 = 3.

However, rank(T) cannot exceed 2, the dimension of the target space.

※ 8

Using the formula $B = S^{-1}AS$,

a) $[T]_\beta = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} = \frac{1}{-1} \begin{bmatrix} 1 & -1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ -2 & -2 \end{bmatrix}$.

b) $[T]_\beta = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{bmatrix} = \begin{bmatrix} 3 & -3 & 1 \\ -3 & 5 & -2 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{bmatrix}$

$$= \begin{bmatrix} 1 & -6 & -18 \\ 2 & 15 & 36 \\ -1 & -6 & -14 \end{bmatrix}$$

※9.

Note that $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 - x_2 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ implies $A = \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix}$.

Put $S = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Using the formula $S[T]_{\beta} = AS$, we have

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 3 & 1 \\ -2 & 0 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\begin{bmatrix} 3a-2b & a \\ 3c-2d & c \end{bmatrix} = \begin{bmatrix} 2a-c & 2b-d \\ c & d \end{bmatrix}.$$

$$\begin{cases} 3a-2b = 2a-c \\ a = 2b-d \\ 3c-2d = c \\ c = d \end{cases} \Rightarrow \begin{cases} a = 2b-c \\ c = d. \end{cases}$$

Find any $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ which satisfy $\begin{cases} a = 2b-c \\ c = d \end{cases}$ and $ad - bc \neq 0$.
($\because$ S is invertible)

An example is $\begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$. In this case, the corresponding basis $\mathcal{B}$ is $\left\{\begin{bmatrix} 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix}\right\}$.

※10.

Denote $B = S^{-1}AS$. By assumption, B has third column $\vec{0}$.

Consider $SB = AS$.

3rd Column of $SB = S \cdot (\text{3rd Column of } B) = S \cdot \vec{0} = \vec{0}$.

Hence,

$\vec{0} =$ 3rd Column of $AS = A \cdot (\text{3rd Column of } S)$.

It follows that 3rd Column of S is in $\ker(A) = \operatorname{Span}\left(\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}\right)$.

An example of such invertible matrix S is $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$.