Mathematics 201, Spring 2017: Assignment #5

Due: **In lecture, Friday, Mar. 17**

Instructions: Please ensure your name, your TA's name and your section number appear on the first page. Also that your answers are legible and all pages are stapled. Page numbers refer to the course text.

Problem #1. Consider the linear transformation $P : \mathbb{R}^2 \to \mathbb{R}^2$ given by orthogonal projection onto the line $L = \{y = 2x\}$. Determine a basis, $\mathcal{B}$, of $\mathbb{R}^2$ so that $[P]_{\mathcal{B}}$ is diagonal.

Problem #2. Let $J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

a) Verify that $J^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -I_2$, $J^3 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = -J$ and $J^4 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$.

b) Show that $J^n = J^{n-4}$ and use this to calculate J^{11}.

Problem #3. Let $S = \begin{bmatrix} 2 & -3 \\ 1 & -1 \end{bmatrix}$, $A = \begin{bmatrix} -5 & 13 \\ -2 & 5 \end{bmatrix}$ and $J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ be from Problem 2.

a) Use S to verify that A is similar to J.

b) Use this fact to compute A^7.

Problem #4. Let $J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ be from Problem 2. Show that J is not similar to any diagonal matrix.

Problem #5. Suppose that A is 2×2 matrix, $A \neq 0_{2\times 2}$ and $A^2 = 0_{2\times 2}$.

a) Show that $\ker(A) = \operatorname{im}(A)$.

b) Show that there is a basis, $\vec{v}_1, \vec{v}_2$, of $\mathbb{R}^2$ that satisfies $A\vec{v}_1 = \vec{0}$ and $A\vec{v}_2 = \vec{v}_1$.

c) Use the basis $\vec{v}_1, \vec{v}_2$ to show that A is similar to $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

Problem #6. Show that the set of all 2×2 matrices, S, that satisfy

$$S \begin{bmatrix} 3 & 2 \\ 4 & 5 \end{bmatrix} = S$$

is a subspace of $\mathbb{R}^{2\times 2}$ and determine its dimension.

Problem #7. Let $\mathbb{R}^{n\times n}$ be the set of $n \times n$ matrices. For $A \in \mathbb{R}^{n\times n}$ show that I_n and A are linearly independent if and only if $A \neq kI_n$ for all $k \in \mathbb{R}$ (i.e., A is not diagonal with all diagonal entries the same).

Problem #8. Consider the map $\operatorname{tr} : \mathbb{R}^{2\times 2} \to \mathbb{R}$ defined by

$$\operatorname{tr} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = a + d.$$

a) Show that tr is a linear map.

b) Find a basis of $\ker(\operatorname{tr})$.

Problem #9. Let P_2 denote the space of polynomials of degree at most 2. Consider the map

$$T : P_2 \to P_2 \text{ defined by } T(p)(x) = p(x+2).$$

Show that this is a linear isomorphism.

Problem #10. Fix a 2×2 matrix $A \in \mathbb{R}^{2\times 2}$. Define a transformation

$$E_A : P_2 \to \mathbb{R}^{2\times 2} \text{ by } E_A(a_0 + a_1 x + a_2 x^2) = a_0 I_2 + a_1 A + a_2 A^2.$$

This may be more compactly written as $E_A(p) = p(A)$.

a) Verify that E_A is a linear transformation.

b) Explain why $\text{rank}(E_A) \geq 1$.

c) Show that if $A \neq kI_2$ for all $k \in \mathbb{R}$, then $\text{rank}(E_A) \geq 2$. (Hint: Use Problem 7).

d) Show that if A is diagonal, then $\text{rank}(E_A) \leq 2$ (Hint: Use the rank-nullity theorem).

Book Problems.

a) Section 3.4: #30, #44, #62
b) Section 4.1: #2, #10, #26, #38
c) Section 4.2: #6, #8, #44