Mathematics 201, Spring 2017: Assignment #6

Due: **In lecture, Friday, Mar. 31**

Instructions: Please ensure your name, your TA's name and your section number appear on the first page. Also that your answers are legible and all pages are stapled. Page numbers refer to the course text.

Problem #1. Fix four data points $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ and (x_4, y_4) with $x_1 < x_2 < x_3 < x_4$.

a) Explain why there is a unique polynomial $p \in P_3$ so that $p(x_i) = y_i$ for $i = 1, \cdots, 4$. Hint: Use the linear transformation $T : P_3 \to \mathbb{R}^4$ given by $T(p) = \begin{bmatrix} p(x_1) \\ p(x_2) \\ p(x_3) \\ p(x_4) \end{bmatrix}$.

b) Find a basis, $\mathcal{B} = (p_1, \ldots, p_4)$, of P_3 so that $T(p_i) = \vec{e}_i$ (T defined above).

c) Use this basis to find the $p \in P_3$ so that $p(-1) = p(1) = 0$ and $p(-2) = p(2) = 1$.

Problem #2. Consider the polynomials $f(x) = x - 1$ and $g(x) = (x+1)(x-k)$ where $k \in \mathbb{R}$ is an arbitrary constant. Determine which values of k make $f(x), xf(x), g(x)$ a basis of P_2.

Problem #3. Let $V = \text{span}(\cos(x), \sin(x)) = \{c_1 \cos(x) + c_2 \sin(x) : c_1, c_2 \in \mathbb{R}\} \subset C^\infty$. Determine values $a, b \in \mathbb{R}$ so that the map $T : V \to V$ given by $T(f) = f'' + af' + bf$ is an isomorphism.

Problem #4. Consider $T : P_2 \to P_2$ given by $T(p)(x) = (x+1)p'(x)$ (e.g., so $T(x) = x + 1$).

a) Determine constants $c_1, c_2, c_3 \in \mathbb{R}$ so that $\mathcal{U} = (1, x + c_1, x^2 + c_2 x + c_3)$ is a basis of P_2 for which $[T]_{\mathcal{U}}$ is diagonal.

b) Compute the change of basis matrix $S_{\mathcal{B}\to\mathcal{U}}$ where $\mathcal{B} = (1, x, x^2)$ is the standard basis of P_2.

c) Verify that the expected relationship between $[T]_{\mathcal{B}}, [T]_{\mathcal{U}}$ and $S_{\mathcal{B}\to\mathcal{U}}$ holds.

Problem #5. Consider the linear transformation $T : \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$ given by

$$T(A) = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} A.$$

Determine $[T]_{\mathcal{U}}$, the matrix of T with respect to the basis,

$$\mathcal{U} = \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right)$$

and use this to determine a a basis of the image and kernel of T.

Problem #6. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation and $\mathcal{U}$ the basis of $\mathbb{R}^{2\times 2}$ from Question 5.

a) Determine $[T]_{\mathcal{B}}$, the matrix of T with respect to the standard basis

$$\mathcal{B} = \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right).$$

b) Determine $S_{\mathcal{B}\to\mathcal{U}}$ the change of basis matrix between $\mathcal{B}$ and $\mathcal{U}$

c) Verify the expected relationship between $[T]_{\mathcal{B}}, [T]_{\mathcal{U}}$ and $S_{\mathcal{B}\to\mathcal{U}}$ holds,

Problem #7. Let V be the subspace of $\mathbb{R}^3$ described by the plane $x_1 + 2x_2 - x_3 = 0$.

a) Determine the change of basis matrix $S_{\mathcal{U}\to\mathcal{B}}$ between the bases of V given by

$$\mathcal{B} = \left(\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right) \text{ and } \mathcal{U} = \left(\begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \right).$$

b) Determine $[T]_{\mathcal{B}}$ and $[T]_{\mathcal{U}}$ for $T : V \to V$ given by

$$T(\vec{x}) = \begin{bmatrix} 3 & 0 & -1 \\ -1 & -1 & 1 \\ 1 & -2 & 1 \end{bmatrix}.$$

c) Verify that the expected relationship between $[T]_{\mathcal{B}}$, $[T]_{\mathcal{U}}$ and $S_{\mathcal{B}\to\mathcal{U}}$ holds.

Problem #8. Consider the vector $\vec{v} = \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}$. Find a basis of $\text{span}(\vec{v})^\perp$ the space of all vectors in $\mathbb{R}^3$ perpendicular to $\vec{v}$.

Problem #9. Consider the subspace, $V \subset \mathbb{R}^3$ given by the plane $x_1 + x_2 - 4x_3 = 0$.

a) Determine constants $c_1, c_2 \in \mathbb{R}$ so that $\vec{u}_1 = c_1 \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$ and $\vec{u}_2 = c_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ is an orthonormal basis of V.

b) Find the matrix, $[\text{proj}_V]$, of the linear transformation given by orthogonal projection onto V.

Problem #10. Consider the linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^2$ given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \right) = \begin{bmatrix} x_1 - 2x_2 + x_3 + 3x_4 \\ x_1 + x_4 \end{bmatrix}.$$

Find a basis of $\ker(T)^\perp$, the space vectors perpendicular to the kernel of T. Hint: First find a basis of $\ker(T)$.

Book Problems.

a) Section 4.2: #22, #54, #64
b) Section 4.3: #2, #8, #42, #60
c) Section 5.1: #4, #10, #16