Mathematics 201, Spring 2017: Assignment #8

Due: **In lecture, Friday, Apr. 21st**

Instructions: Please ensure your name, your TA's name and your section number appear on the first page. Also that your answers are legible and all pages are stapled. Page numbers refer to the course text.

Problem #1. Find the least squares solutions to the following system $A\vec{x} = \vec{b}$ where

a) $A = \begin{bmatrix} 2 & 1 \\ 0 & -1 \\ 1 & 2 \end{bmatrix}, \vec{b} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

b) $A = \begin{bmatrix} 0 & -1 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \vec{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

Problem #2. Let $A = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$. Determine whether the least square solution to $A\vec{x} = \vec{e}_1$ or to $A\vec{x} = \vec{e}_2$ is closer to a true solution.

Problem #3. Find the quadratic polynomial that fits the data $(0,0), (2,1), (1,1)$ and $(-2,0)$ the best in the sense of least squares. Sketch the solution.

Problem #4.

a) Show that if $A \in \mathbb{R}^{n\times m}$ and $B \in \mathbb{R}^{m\times n}$ then $\operatorname{tr}(AB) = \operatorname{tr}(BA)$.

b) Show that if $Q \in \mathbb{R}^{n\times n}$ is orthogonal, and $A, B \in \mathbb{R}^{n\times n}$, then

$$\langle A, B\rangle_{HS} = \langle QA, QB\rangle_{HS} = \langle AQ, BQ\rangle_{HS}$$

Here $\langle A, B\rangle_{HS} := \operatorname{tr}(A^\top B)$ is the Hilbert-Schmidt inner product.

Problem #5. Let $A \in \mathbb{R}^{n\times n}$ have rows $\vec{a}_1^\top, \dots, \vec{a}_n^\top$. Use that $A\vec{x} = \begin{bmatrix} \vec{a}_1 \cdot \vec{x} \\ \vdots \\ \vec{a}_n \cdot \vec{x} \end{bmatrix}$ and the Cauchy-Schwarz inequality to show that for any $\vec{x} \in \mathbb{R}^n$,

$$||A\vec{x}|| \le ||A||_{HS}||\vec{x}||$$

where here $||A||_{HS} = \sqrt{\langle A, A\rangle_{HS}}$. Give a non-zero example of an A and $\vec{x}$ where you have equality.

Problem #6. For what values of $a, b, c \in \mathbb{R}$ is the following an inner product on $\mathbb{R}^2$

$$\left\langle \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}, \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \right\rangle = ax_1x_2 + bx_1y_2 + cx_2y_1 + y_1y_2.$$

Problem #7. Use the determinant to determine values k for which the given matrix is invertible.

a) $\begin{bmatrix} 1 & k \\ k & 9 \end{bmatrix}$

b) $\begin{bmatrix} k & 3 & k \\ 0 & 2 & -k \\ 0 & 0 & k+1 \end{bmatrix}$.

Problem #8. Suppose $A = \begin{bmatrix} \vec{v}_1 & | & \vec{v}_2 & | & \vec{v}_3 & | & \vec{v}_4 \end{bmatrix}$ where $\vec{v}_i \in \mathbb{R}^4$. If $\det(A) = -4$ determine

a) $\det \begin{bmatrix} \vec{v}_3 & | & 2\vec{v}_1 & | & \vec{v}_4 & | & \vec{v}_2 \end{bmatrix}$

b) $\det \begin{bmatrix} \vec{v}_1 - \vec{v}_2 + \vec{v}_3 & | & \vec{v}_2 + \vec{v}_3 & | & \vec{v}_4 & | & \vec{v}_3 \end{bmatrix}$.

Problem #9. Let $A \in \mathbb{R}^{n\times n}$. Determine $\det(kA)$ in terms of k and $\det(A)$.

Problem #10. Use the determinant to show that if n is odd and $A \in \mathbb{R}^{n\times n}$ is skew-symmetric, then A is not invertible. (Hint: Use answer to previous question).

Book Problems.

a) Section 5.4 #8, #20, #30
b) Section 5.5 #16, #20, #24
c) Section 6.1 #28, #54
d) Section 6.2 #6, #18