1. The least-square solution of $A\vec{x} = \vec{b}$ are the exact solutions of the system $A^T A\vec{x} = A^T\vec{b}$

   (a)

$$A^T A = \begin{bmatrix} 2 & 0 & 1 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & -1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 4 \\ 4 & 6 \end{bmatrix}$$

$$A^T\vec{b} = \begin{bmatrix} 2 & 0 & 1 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

   So $A^T A\vec{x} = A^T\vec{b}$ gives us:

$$\begin{bmatrix} 5 & 4 \\ 4 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

   Therefore:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{5}{7} \\ \frac{-9}{14} \end{bmatrix}$$

   (b)

$$A^T A = \begin{bmatrix} 0 & 2 & 1 \\ -1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 5 & 3 & 2 \\ 3 & 3 & 0 \\ 2 & 0 & 2 \end{bmatrix}$$

$$A^T\vec{b} = \begin{bmatrix} 0 & 2 & 1 \\ -1 & -1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$$

   So $A^T A\vec{x} = A^T\vec{b}$ gives us:

$$\begin{bmatrix} 5 & 3 & 2 \\ 3 & 3 & 0 \\ 2 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$$

   Note that $I = II + III$ so we can simplify our matrix to:

$$\begin{bmatrix} 0 & 0 & 0 \\ 3 & 3 & 0 \\ 2 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

   This gives us:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ \frac{1}{3} - x_1 \\ 1 - x_1 \end{bmatrix}$$

   So our final answer is:

$$\vec{x} = \begin{bmatrix} t \\ \frac{1}{3} - t \\ 1 - t \end{bmatrix}$$

2. Note that $A^T A\vec{x}^8 = A^T \vec{e_1}$ gives us that:

$$\begin{bmatrix} 5 & 1 & 3 \\ 1 & 2 & 1 \\ 3 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

So,

$$\vec{x}^*_{e_1} = \begin{bmatrix} \frac{1}{2} \\ \frac{-1}{5} \\ \frac{1}{10} \end{bmatrix}$$

and $||\vec{e_1} - A\vec{x}^*_{e_1}|| = \frac{\sqrt{10}}{10}$. While

$$\vec{x}^*_{e_2} = \begin{bmatrix} 0 \\ \frac{2}{5} \\ \frac{1}{5} \end{bmatrix}$$

and $||\vec{e_2} - A\vec{x}^*_{e_2}|| = \frac{\sqrt{10}}{5}$.
So the least-square solution to $A\vec{x} = \vec{e_1}$ is closer to a true solution.

3. Since we know we have a quadratic, our answer must be in the form: $f(x) = ax^2 + bx + c$. So the points (0,0), (2,1), (1,1), and (-2, 0) give us the system of equations:

$$\begin{aligned} a*0+b*0+c &= 0 \\ a*4+b*2+c &= 1 \\ a*1+b*1+c &= 1 \\ a*4+b*-2+c &= 0 \end{aligned}$$

This becomes the matrix:

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 4 & 2 & 1 \\ 1 & 1 & 1 \\ 4 & -2 & 1 \end{bmatrix}$$

and the vector

$$b = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

So $A^T A\vec{x} = A^T \vec{b}$ gives us:

$$\begin{bmatrix} 33 & 1 & 9 \\ 1 & 9 & 1 \\ 9 & 1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ 2 \end{bmatrix}$$

Solving this system of equations gives us the polynomial:
$f(x) = \frac{3}{44}x^2 + \frac{13}{44}x + \frac{3}{11}$

4. (a) Note that each entry in A can be written as (a_{ij}) and every entry in B can be written as (b_{ij}). Now note that the product of AB is:

$$(AB)_{ij} = \sum_{k=1}^{n} a_{ik} b_{ki}.$$

Then since the trace of a matrix is the sum of its diagonal entries:

$$Tr(AB) = \sum_{j=1}^{n} \sum_{k=1}^{n} a_{jk} b_{kj}.$$

Now we want to show that BA yields the same result.

$$(BA)_{ij} = \sum_{k=1}^{n} b_{ik} a_{ki}.$$

and

$$Tr(BA) = \sum_{j=1}^{n} \sum_{k=1}^{n} b_{jk} a_{kj}.$$

Thus we see that $Tr(AB) = Tr(BA)$

(b) Given that Q is orthogonal we know that $Q^T = Q^{-1}$ We essentially want to prove:

$$Tr(A^T B) = Tr((QA)^T QB) = Tr((AQ)^T BQ)$$

Let's start with the second term:

$$Tr((QA)^T QB) = Tr(A^T Q^T QB) = Tr(A^T Q^{-1} QB) Tr(A^T B)$$

so, the first part of the equation is true. Evaluating the third term we get:

$$Tr((AQ)^T BQ) = Tr(Q^T A^T BQ) = Tr(BQQ^T A^T) = Tr(BA^T)$$

Thus:

$$Tr(A^T B) = Tr((QA)^T QB) = Tr((AQ)^T BQ)$$

5. We know that $||A||_{HS} = \sqrt{tr(A^T A)}$ and

$$tr(A^T A) = \sum_{i=1}^{n} a_i^2$$

. Given that matrix A is made up of the vectors $\vec{a_i}$ Thus we see that $||A||_{HS} = \sqrt{a1^2 + a_2^2 + ... + a_n^2}$
So we get

$$||A||_{HS} ||\vec{x}||_{HS} = \sqrt{a_1^2 + a_2^2 + ... + a_n^2} ||\vec{x}||_{HS}$$

and

$$||A\vec{x}|| = \sqrt{\sum_{i=1}^{n} (a_i \vec{x})^2}$$

By Cauchy-Schwartz Inequality we have:

$$|a_i \vec{x}| \leq ||a_i|| ||\vec{x}||$$

Thus,

$$||A\vec{x}|| \leq ||A||_{HS} ||\vec{x}||$$

6. To prove this is an inner product we need to show that:
   1. $< f, g >=< g, f >$ which holds true if b =c.
   2. $< f + h, g >=< f, g > + < hg >$ which holds true always.
   3. $< kf, g >= k < f, g >$ which holds true always.
   4. $< f, f >> 0$ which holds true if $b^2 < a$.

7. (a)

   $$\det \begin{bmatrix} 1 & k \\ k & 9 \end{bmatrix} = 9 - k^2$$

   In order for the matrix to be invertible the determinant cannot equal zero. Thus k can take on any value except $\pm$ 3.

   (b)

   $$\det \begin{bmatrix} k & 3 & k \\ 0 & 2 & -k \\ 0 & 0 & k+1 \end{bmatrix} = k(2)(k+1)$$

   In order for the matrix to be invertible the determinant cannot equal zero. Thus k can take on any value except 0 or -1.

8. (a) The determinant will increase by 2 since the vector $\vec{v_1}$ is increased by two and since we switched the rows 3 times, the determinant will be multiplied by $(-1)^3$. So det = 8.

   (b) Adding the vectors will have no effect on the determinant so the only thing we need to take into account is the switching of $\vec{v}_3$ and $\vec{v}_4$. So det = 4.

9. det(kA) = k^ndet(A)

10. If A is skew-symmetric then:

    $$det(A) = det(A^T) = det(-A) = (-1)^n det(A)$$

    If n is an odd number then $(-1)^n = -1$ so in order for det(A) = -det(A), det(A) = 0. Therefore, A is not invertible.