Mathematics 201, Spring 2017: Assignment #9

Due: **In lecture, Friday, Apr. 28th**

Instructions: Please ensure your name, your TA's name and your section number appear on the first page. Also that your answers are legible and all pages are stapled. Page numbers refer to the course text.

Problem #1. Calculate the determinant of the following matrices by using any method you like.

a) $\begin{bmatrix} 0 & 2 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 3 & 0 \end{bmatrix}$ b) $\begin{bmatrix} 2 & 2 & -4 \\ 0 & 1 & 1 \\ 1 & 1 & 3 \end{bmatrix}$

Problem #2. Compute the determinant of the following linear transformations:

a) $T : P_2 \to P_2$ defined by $T(p)(x) = xp'(x) - p(x)$

b) $T : \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$ defined by $T(A) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} A$.

Problem #3. Fix a vector $\vec{v} \in \mathbb{R}^n$.

a) Find a basis $\mathcal{B}$, so that $[I_n + \vec{v}\vec{v}^\top]_\mathcal{B}$ is diagonal and the non-zero entries are 1 or $1 + ||\vec{v}||^2$. (Hint: When $\vec{v} \neq 0$, consider $\vec{v}$ together with a basis of $\operatorname{span}(\vec{v})^\perp$.)

b) Conclude that $\det\left(I_n + \vec{v}\vec{v}^\top\right) = 1 + ||\vec{v}||^2$.

Problem #4. Suppose $J \in \mathbb{R}^{n\times n}$ satisfies $J^2 = -I_n$. Use the determinant, to show that $n = 2m$ is even.

Problem #5. Let $M = \begin{bmatrix} \vec{v}_1 & | & \vec{v}_2 & | & \vec{v}_3 \end{bmatrix} \in \mathbb{R}^{3\times 3}$.

a) Show that $|\det(M)| \leq ||\vec{v}_1||||\vec{v}_2||||\vec{v}_3||$. (Hint: Use the QR factorization).

b) Give an example of an invertible matrix for which equality is achieved.

Problem #6. Let $P \subset \mathbb{R}^3$ be the parallelepiped spanned by $\begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

a) Compute the volume of P. b) Compute the surface area of P.

Problem #7. Determine all $A \in \mathbb{R}^{2\times 2}$ for which $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ is an eigenvector with associated eigenvalue 2.

Problem #8. Find the eigenvalues and their algebraic multiplicities for the following matrices

a) $\begin{bmatrix} 0 & -4 \\ 1 & -4 \end{bmatrix}$ b) $\begin{bmatrix} -3 & 0 & 4 \\ 0 & -1 & 0 \\ -2 & 7 & 3 \end{bmatrix}$.

Problem #9. Find all eigenvalues of the following 2×2 matrices

a) $\begin{bmatrix} a & b \\ b & -a \end{bmatrix}$ for $a, b \in \mathbb{R}$.

b) $\begin{bmatrix} a & b \\ b & a \end{bmatrix}$ for $a, b \in \mathbb{R}$.

Problem #10. Show that if A is a symmetric matrix and $\vec{v}_1, \vec{v}_2$ are eigenvectors of A with different associated eigenvalues, then $\vec{v}_1$ is orthogonal to $\vec{v}_2$. (Hint: Use that $(A\vec{x}) \cdot \vec{y} = \vec{x} \cdot A^\top \vec{y}$.)

Book Problems.

a) Section 6.2 #30, #46
b) Section 6.3 #2, #6, #14
c) Section 7.1 #6, #8
d) Section 7.2 #2, #32, #34