Problem #1. Calculate the determinant of the following matrices by using any method you like.

$$\text{a)} \begin{bmatrix} 0 & 2 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 3 & 0 \end{bmatrix} \qquad \text{b)} \begin{bmatrix} 2 & 2 & -4 \\ 0 & 1 & 1 \\ 1 & 1 & 3 \end{bmatrix}$$

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a) $\begin{vmatrix} 0 & 2 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 3 & 0 \end{vmatrix}$

$= -\begin{vmatrix} -1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 3 & 0 \end{vmatrix}$ By row-swap

$= \begin{vmatrix} -1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 2 \end{vmatrix}$ By row-swap

$= -1 \times 2 \times 3 \times 2 = -12$

b) $\begin{vmatrix} 2 & 2 & -4 \\ 0 & 1 & 1 \\ 1 & 1 & 3 \end{vmatrix}$

$= \begin{vmatrix} 2 & -4 \\ 1 & 3 \end{vmatrix} - \begin{vmatrix} 2 & 2 \\ 1 & 1 \end{vmatrix}$ By expanding out on the 2nd row

$= 10 - 0 = 10$

Problem #2. Compute the determinant of the following linear transformations:

a) $T : P_2 \to P_2$ defined by $T(p)(x) = xp'(x) - p(x)$

b) $T : \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$ defined by $T(A) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} A$.

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a) Let $\beta = \{1, x, x^2\}$, we'll solve for $[T]_\beta$.

$T(a + bx + cx^2) = x(b + 2cx) - (a + bx + cx^2) = -a + cx^2$

$$\implies [T]_\beta = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\implies \det(T) = \det([T]_\beta) = 0$$

b) Let $\beta = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$.

Then, $[T]_\beta = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 3 & 4 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 3 & 4 \end{bmatrix}$ by inspection.

$$\implies \det(T) = \det([T]_\beta) = \det\left(\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\right) \cdot \det\left(\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\right)$$

$$= (-2)^2 = 4$$

Problem #3. Fix a vector $\vec{v} \in \mathbb{R}^n$.

a) Find a basis β, so that $[I_n + \vec{v}\vec{v}^T]_\beta$ is diagonal and the non-zero entries are 1 or $1 + \|\vec{v}\|^2$. (Hint: When $\vec{v} \neq 0$, consider $\vec{v}$ together with a basis of $\text{span}(v)^\perp$.)

b) Conclude that $\det(I_n + \vec{v}\vec{v}^T) = 1 + \|\vec{v}\|^2$.

If $\vec{v} = \vec{0}$, then $I_n + \vec{v}\vec{v}^T = I_n$, just take β as the standard basis and the result is obvious. So now let $\vec{v} \neq \vec{0}$.

Following the hint, let $\{\vec{w}_1, \vec{w}_2, ..., \vec{w}_{n-1}\}$ be a basis of $\text{span}(v)^\perp$. I claim that $\beta = \{\vec{v}, \vec{w}_1, \vec{w}_2, ..., \vec{w}_{n-1}\}$. We'll solve for $[I_n + \vec{v}\vec{v}^T]_\beta$ column-by-column.

$$(I_n + \vec{v}\vec{v}^T)\vec{v}$$

$$= \vec{v} + \vec{v}(\vec{v} \cdot \vec{v})$$

$$= (1 + \vec{v} \cdot \vec{v})\vec{v}$$

$$= (1 + \|\vec{v}\|^2)\vec{v}$$

$$(I_n + \vec{v}\vec{v}^T)\vec{w}_i$$

$$= \vec{w}_i + \vec{v}(\vec{v} \cdot \vec{w}_i)$$

$$= \vec{w}_i$$

because $\vec{v} \cdot \vec{w} = 0$. They're perpendicular to each others because $\vec{w}_i \in \text{span}(v)^{\perp}$ by assumption.

$$\implies [I_n + \vec{v}\vec{v}^T]_\beta = \begin{bmatrix} 1 + \|\vec{v}\|^2 & 0 & 0 & 0 & & 0 \\ 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & & 0 \\ 0 & 0 & 0 & 1 & & 0 \\ & \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 \end{bmatrix}$$

Clearly, $\det(I_n + \vec{v}\vec{v}^T) = \det([I_n + \vec{v}\vec{v}^T]_\beta) = 1 + \|\vec{v}\|^2$.

Problem #4. Suppose $J \in (R)^{n \times n}$ satisfies $J^2 = -I_n$. Use the determinant, to show that $n = 2m$ is even.

$\det(J^2)$

$= \det(JJ)$

$= \det(J) \cdot \det(J)$

$= \det(J)^2$

$\det(-I_n) = (-1)^n$, because $\det(kA) = k^n \cdot \det(A)$.

$\implies \det(J)^2 = (-1)^n$

Left hand side is nonnegative, so n can't be odd.

Problem #5. Let $M = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{bmatrix} \in \mathbb{R}^{3 \times 3}$.

a) Show that $|\det(M) \leq \|\vec{v}_1\|\|\vec{v}_2\|\|\vec{v}_3\|$. Hint: Use the QR factorization).

b) Give an example of an invertible matrix for which equality is achieved.

Following the hint, let $M = QR$. Taking the determinant on both sides, we have $\det(M) = \det(QR) = \det(Q) \cdot \det(R) = \pm\det(R)$ (Recall that the determinant of an orthogonal matrice is ± 1). Now, let $Q = \begin{bmatrix} \vec{w}_1 & \vec{w}_2 & \vec{w}_3 \end{bmatrix}$.

Then, $R = Q^{-1}M = Q^T M$

$$= \begin{bmatrix} \vec{w}_1 \\ \vec{w}_2 \\ \vec{w}_3 \end{bmatrix} \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{bmatrix}$$

$= \begin{bmatrix} \vec{w}_1 \cdot \vec{v}_1 & \vec{w}_1 \cdot \vec{v}_2 & \vec{w}_1 \cdot \vec{v}_3 \\ 0 & \vec{w}_2 \cdot \vec{v}_2 & \vec{w}_2 \cdot \vec{v}_3 \\ 0 & 0 & \vec{w}_3 \cdot \vec{v}_3 \end{bmatrix}$ Upper-triangular by properties of QR factorization

$\implies \det(R) = (\vec{w}_1 \cdot \vec{v}_1)(\vec{w}_2 \cdot \vec{v}_2)(\vec{w}_3 \cdot \vec{v}_3)$

$\leq \|\vec{v}_1\|\|\vec{v}_2\|\|\vec{v}_3\|$, by Cauchy-Schwarz Inequality. Recall that $\|w_i\| = 1$ by properties of orthogonal matrices.

An example where the equality is achieved is I_3.

Problem #6. Let $P \subset \mathbb{R}^3$ be the parallelpiped spanned by $\begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

$$V = \left| \det \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix} \right|$$

$$= \left| \det \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} \right|$$

$$= 2$$

$$SA = 2|\vec{v}_1 \times \vec{v}_2| + 2|\vec{v}_1 \times \vec{v}_3| + 2|\vec{v}_2 \times \vec{v}_3|$$

$$= 2\left\| \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} \times \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\| + 2\left\| \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\| + 2\left\| \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\|$$

$$= 2\left\| \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix} \right\| + 2\left\| \begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix} \right\| + 2\left\| \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right\|$$

$$= 2\sqrt{14} + 2 \cdot 2 + 2\sqrt{2}$$

$$= 4 + 2\sqrt{2} + 2\sqrt{14}$$

Problem #7. Determine all $A \in \mathbb{R}^{2\times 2}$ for which $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ is an eigenvector with associated eigenvalue 2.

Solution 1:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix} = 2 \cdot \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$a - 2b = 2 \implies a = 2b + 2$$

$$c - 2d = -4 \implies c = 2d - 4$$

$$\implies A = \begin{bmatrix} 2b+2 & b \\ 2d-4 & d \end{bmatrix}, \forall b, d \in \mathbb{R}$$

Solution 2:

Let $\beta = \left\{ \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$

Then, $[A]_\beta = \begin{bmatrix} 2 & a \\ 0 & b \end{bmatrix}$

So $A = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 2 & a \\ 0 & b \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}^{-1}$

$$= \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 2 & a \\ 0 & b \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 2+2a & a \\ 2b & b \end{bmatrix}$$

$$= \begin{bmatrix} 2+2a & a \\ -4-4a+2b & -2a+b \end{bmatrix}, \forall a, b \in \mathbb{R}$$

Problem #8. Find the eigenvalues and their algebraic multiplicities for the following matrices

$$\text{a) } \begin{bmatrix} 0 & -4 \\ 1 & -4 \end{bmatrix} \qquad \text{b) } \begin{bmatrix} -3 & 0 & 4 \\ 0 & -1 & 0 \\ -2 & 7 & 3 \end{bmatrix}$$

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a) $\lambda^2 - \text{tr}A\lambda + \det A = 0$

$\lambda^2 + 4\lambda + 4 = 0$

$\lambda = -2$ has multiplicity 2.

b) $\begin{vmatrix} -3-\lambda & 0 & 4 \\ 0 & -1-\lambda & 0 \\ -2 & 7 & 3-\lambda \end{vmatrix} = 0$

$\implies (-1-\lambda)\begin{vmatrix} 3-\lambda & 4 \\ -2 & 3-\lambda \end{vmatrix} = 0$

$\implies (-1-\lambda)((-3-\lambda)(3-\lambda)+8) = 0$

$\implies (-1-\lambda)(\lambda^2-1) = 0$

$\implies -(\lambda+1)^2(\lambda-1) = 0$

So $\lambda = -1$ has multiplicity 2, $\lambda = 1$ has multiplicity 1.

Problem #9. Find all eigenvalues of the following 2×2 matrices

a) $\begin{bmatrix} a & b \\ b & -a \end{bmatrix}$ for $a, b \in \mathbb{R}$ b) $\begin{bmatrix} a & b \\ b & a \end{bmatrix}$ for $a, b \in \mathbb{R}$

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a) $\lambda^2 - \text{tr}A\lambda + \det A = 0$

$\implies \lambda^2 - a^2 - b^2 = 0$

$\implies \lambda = \pm\sqrt{a^2+b^2}$

b) $\lambda^2 - \text{tr}A\lambda + \det A = 0$

$\implies \lambda^2 - 2a\lambda + a^2 - b^2 = 0$

$\implies (\lambda - a)^2 = b^2$

$\implies \lambda - a = \pm b$

$\implies \lambda = a \pm b$

Problem #10. Show that if A is a symmetric matrix and $\vec{v}_1, \vec{v}_2$ are eigenvectors of A with different associated eigenvalues, then $\vec{v}_1$ is orthogonal to $\vec{v}_2$. (Hint: Use that $(A\vec{x}) \cdot \vec{y} = \vec{x} \cdot (A^T\vec{y})$.)

$(A\vec{x}) \cdot \vec{y} = \vec{x} \cdot (A^T\vec{y}) = \vec{x} \cdot (A\vec{y})$

$\implies (\lambda_1\vec{x}) \cdot \vec{y} = \vec{x} \cdot (\lambda_2\vec{y})$

$\implies \lambda_1(\vec{x} \cdot \vec{y}) = \lambda_2(\vec{x} \cdot \vec{y})$

$\implies \vec{x} \cdot \vec{y} = 0$, otherwise we could cancel it on both sides and get $\lambda_1 = \lambda_2$.

So $\vec{x}$ is orthogonal to $\vec{y}$.