LINEAR ALGEBRA
First Midterm Exam

JOHNS HOPKINS UNIVERSITY
SPRING 2013

You have 50 MINUTES.
No calculators, books or notes allowed.

Academic Honesty Certificate. I agree to complete this exam without unauthorized assistance from any person, materials or device.

Signature: ______________________ Date: __________

Name: ______________________ Section №: __________
(or TA's name)

Question	*Score*
1	25
2	25
3	25
4	25
5 (bonus)	20

120 ← I hope you get this ^^

(1) (a) [15 points] Find all solutions to the system of equations:

$$\begin{cases} x + y + 6z = 8 \\ 2x + 3y + 16z = 21 \end{cases}$$

using Gaussian elimination. Is the system consistent? Why?

$$\left[\begin{array}{ccc|c} 1 & 1 & 6 & 8 \\ 2 & 3 & 16 & 21 \end{array}\right] -2\mathrm{I} \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 6 & 8 \\ 0 & 1 & 4 & 5 \end{array}\right] -\mathrm{II} \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 2 & 3 \\ 0 & 1 & 4 & 5 \end{array}\right]$$

Yes, the system is consistent because there is no "$0=1$" in the system corresponding to the rref.

Solution of the system is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2s+3 \\ -4s+5 \\ s \end{bmatrix} \quad \text{where } s \text{ is arbitrary.}$$

(b) [10 points] Does the system of equations:

$$\begin{cases} 2x + 12z = 14 \\ 2y + 16z = 18 \\ x + 2y + 22z = 25 \end{cases}$$

have a unique solution? Justify your answer.

$$\left[\begin{array}{ccc|c} 2 & 0 & 12 & 14 \\ 0 & 2 & 16 & 18 \\ 1 & 2 & 22 & 25 \end{array}\right] \div 2 \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 6 & 7 \\ 0 & 2 & 16 & 18 \\ 1 & 2 & 22 & 25 \end{array}\right] -\mathrm{I}$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 6 & 7 \\ 0 & 2 & 16 & 18 \\ 0 & 2 & 16 & 18 \end{array}\right] \div 2 \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 6 & 7 \\ 0 & 1 & 8 & 9 \\ 0 & 2 & 16 & 18 \end{array}\right] -2\mathrm{II}$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 6 & 7 \\ 0 & 1 & 8 & 9 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

No, the system has infinitely many solutions, because there is a free variable z.

(2) [25 points] Let:

$$A = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 5 \\ 1 & 1 & 1 & 15 \end{bmatrix}$$

Can an equation:

$$A\vec{x} = \vec{b}$$

have infinitely many solutions while another equation:

$$A\vec{x} = \vec{c}$$

has none whatsoever? If no, explain why not. If yes, find vectors $\vec{b}$ and $\vec{c}$ in $\mathbf{R}^4$ for which this is true.

$$\begin{array}{cc} \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 5 \\ 1 & 1 & 1 & 15 \end{bmatrix} & \begin{array}{c} \\ \\ \\ -\mathrm{I} \end{array} \end{array} \to \begin{array}{cc} \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 5 \\ 0 & 2 & 1 & 15 \end{bmatrix} & \begin{array}{c} +\mathrm{II} \\ \\ \\ -2\mathrm{II} \end{array} \end{array} \to$$

$$\begin{array}{cc} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 3 & 15 \end{bmatrix} & \begin{array}{c} +\mathrm{III} \\ +\mathrm{III} \\ \\ -3\mathrm{III} \end{array} \end{array} \to \begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Yes. If $\vec{b} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, then the last column of rref $[A \mid \vec{b}]$ is still $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, and the system $A\vec{x} = \vec{b}$ has infinitely many solutions.

Let the last column of rref $[A \mid \vec{c}]$ be $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ so that the system $A\vec{x} = \vec{c}$ is inconsistent. Then go backward, reversing the above Gaussian elimination to find $\vec{c}$:

$$\begin{array}{cc} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} & \begin{array}{c} -\mathrm{III} \\ -\mathrm{III} \\ \\ +3\mathrm{III} \end{array} \end{array} \to \begin{array}{cc} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} & \begin{array}{c} -\mathrm{II} \\ \\ \\ +2\mathrm{II} \end{array} \end{array} \to \begin{array}{cc} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} & \begin{array}{c} \\ \\ \\ +\mathrm{I} \end{array} \end{array} \to \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

If $\vec{c} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$, the system $A\vec{x} = \vec{c}$ has no solution.

(3) (a) [5 points] Compute the matrix products BA and AB where:

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$BA = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$AB = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

(b) [10 points] Does the matrix A above have an inverse? If yes, compute it. If no, why not?

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{matrix} \\ \\ -\mathrm{I} \end{matrix} \rightarrow \begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & -1 & 0 & 1 \end{bmatrix} \begin{matrix} \\ \\ \times(-1) \end{matrix}$$

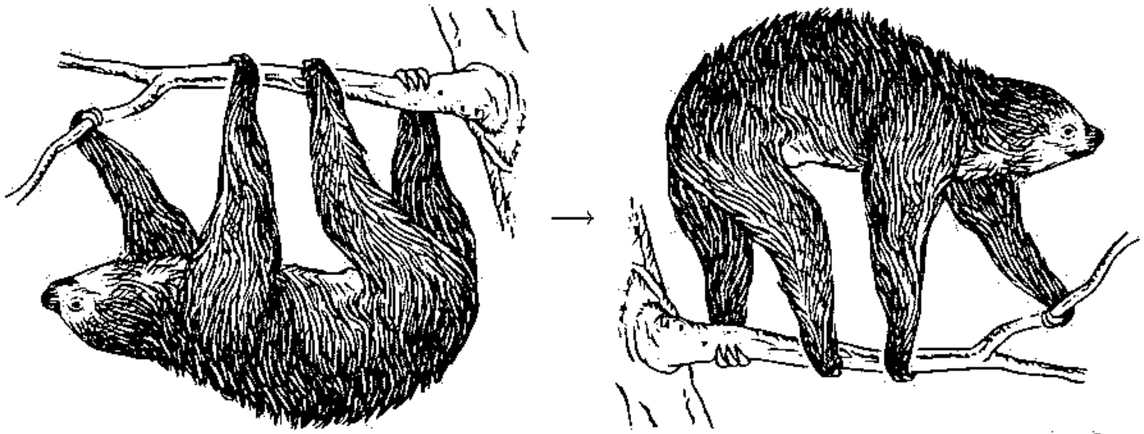
$$\rightarrow \begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & -1 \end{bmatrix} \begin{matrix} -\mathrm{III} \\ \\ \\ \end{matrix} \rightarrow \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & -1 \end{bmatrix}$$

The left half of the last matrix is I_3, so A is invertible,

and $A^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & -1 \end{bmatrix}$.

(3) (c) [10 points] Write down the matrix for the following linear transformation. (The origin is at the center of each drawing.) Explain how you reached your answer.



[This must be how sloths look to one another!]

The linear transformation is rotation clockwise through π.

Its matrix is $\begin{bmatrix} \cos(-\pi) & -\sin(-\pi) \\ \sin(-\pi) & \cos(-\pi) \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$.

(4) (a) [5 points] What does it mean to say that vectors $\vec{v}_1, \ldots, \vec{v}_n$ are *linearly independent?*

A list of vectors $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n$ are linearly independent if the system

$$c_1 \vec{v}_1 + c_2 \vec{v}_2 + \cdots + c_n \vec{v}_n = \vec{0}$$

has only the zero solution $c_1 = c_2 = \cdots = c_n = 0$.

(b) [5 points] Are these vectors linearly independent? ~~Justify your answer using determinants.~~

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad \vec{v}_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

Yes. $c_1 \vec{v}_1 + c_2 \vec{v}_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow c_1 = c_2 = 0$.

(4) (c) [15 points] What is the dimension of the space spanned by the following vectors? Explain your approach and show your work.

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 2 \\ 2 \end{bmatrix} \qquad \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 2 \\ 1 \end{bmatrix} \qquad \vec{v}_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \\ 1 \end{bmatrix} \qquad \vec{v}_4 = \begin{bmatrix} 1 \\ -1 \\ 3 \\ 2 \\ 3 \end{bmatrix}$$

$$A = \begin{array}{c} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 1 & 2 & -1 \\ 2 & 1 & 1 & 3 \\ 2 & 2 & 2 & 2 \\ 2 & 1 & 1 & 3 \end{bmatrix} \end{array} \begin{array}{l} \\ -2\mathrm{I} \\ -2\mathrm{I} \\ -2\mathrm{I} \\ -2\mathrm{I} \end{array} \rightarrow \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & 0 & -3 \\ 0 & -1 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 1 \end{bmatrix} \begin{array}{l} \\ \div(-1) \\ \\ \\ \\ \end{array} \rightarrow \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & -1 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 1 \end{bmatrix} \begin{array}{l} -\mathrm{II} \\ \\ +\mathrm{II} \\ \\ +\mathrm{II} \end{array}$$

$$\rightarrow \begin{bmatrix} 1 & 0 & 1 & -2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & -1 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 4 \end{bmatrix} \begin{array}{l} \\ \\ \div(-1) \\ \\ \\ \end{array} \begin{bmatrix} 1 & 0 & 1 & -2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 4 \end{bmatrix} \begin{array}{l} -\mathrm{III} \\ \\ \\ \\ +\mathrm{III} \end{array} \rightarrow$$

$$\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\dim \operatorname{span} \{ \vec{v}_1, \vec{v}_1, \vec{v}_3, \vec{v}_4 \} = \dim \operatorname{im}(A) = 3.$

(5) [20 bonus points] Find a basis for the image of the linear transformation:

$$A = \begin{bmatrix} a & a & b & a \\ a & a & b & 0 \\ a & b & a & b \\ a & b & a & 0 \end{bmatrix}$$

for any real numbers a and b.

[*Hint: The special cases* $(a,b) = (0,0)$ *and* $(a,b) = (0,1)$ *immediately show that the number of basis vectors will depend on the values* a *and* b *take, so carry out as much row reduction as possible without dividing by possibly vanishing numbers and break into cases at the last step.*]

1. If $a \neq 0$:

$$\begin{bmatrix} a & a & b & a \\ a & a & b & 0 \\ a & b & a & b \\ a & b & a & 0 \end{bmatrix} \begin{matrix} \div a \\ \\ \\ \\ \end{matrix} \longrightarrow \begin{bmatrix} 1 & 1 & \frac{b}{a} & 1 \\ a & a & b & 0 \\ a & b & a & b \\ a & b & a & 0 \end{bmatrix} \begin{matrix} \\ -a\mathrm{I} \\ -a\mathrm{I} \\ -a\mathrm{I} \end{matrix} \longrightarrow$$

$$\begin{bmatrix} 1 & 1 & \frac{b}{a} & 1 \\ 0 & 0 & 0 & -a \\ 0 & b-a & a-b & b-a \\ 0 & b-a & a-b & -a \end{bmatrix} \begin{matrix} \\ \div(-a) \\ \\ \\ \end{matrix} \longrightarrow \begin{bmatrix} 1 & 1 & \frac{b}{a} & 1 \\ 0 & 0 & 0 & 1 \\ 0 & b-a & a-b & b-a \\ 0 & b-a & a-b & -a \end{bmatrix} \begin{matrix} -\mathrm{II} \\ \\ -(b-a)\mathrm{II} \\ +a\mathrm{II} \end{matrix}$$

$$\begin{bmatrix} 1 & 1 & \frac{b}{a} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & b-a & a-b & 0 \\ 0 & b-a & a-b & 0 \end{bmatrix}$$

① If $b-a \neq 0$ $(b \neq a)$:

$$\begin{bmatrix} 1 & 1 & \frac{b}{a} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & b-a & a-b & 0 \\ 0 & b-a & a-b & 0 \end{bmatrix} \begin{matrix} \\ \\ \div(b-a) \\ \\ \end{matrix}$$

$$\rightarrow \begin{bmatrix} 1 & 1 & \frac{b}{a} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & b-a & a-b & 0 \end{bmatrix} \begin{matrix} -\mathrm{III} \\ \\ \\ -(b-a)\mathrm{III} \end{matrix} \rightarrow \begin{bmatrix} 1 & 0 & \frac{b}{a}+1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

basis of image: $\begin{bmatrix} a \\ a \\ a \\ a \end{bmatrix}, \begin{bmatrix} a \\ a \\ b \\ b \end{bmatrix}, \begin{bmatrix} a \\ 0 \\ b \\ 0 \end{bmatrix}.$

② If $b - a = 0$:

$$\begin{bmatrix} 1 & 1 & \frac{b}{a} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

basis of image $\begin{bmatrix} a \\ a \\ a \\ a \end{bmatrix}, \begin{bmatrix} a \\ 0 \\ b \\ 0 \end{bmatrix}.$

2. If $a = 0$:

$$\begin{bmatrix} 0 & 0 & b & 0 \\ 0 & 0 & b & 0 \\ 0 & b & 0 & b \\ 0 & b & 0 & 0 \end{bmatrix}$$

① If $b \neq 0$:

$$\begin{bmatrix} 0 & 0 & b & 0 \\ 0 & 0 & b & 0 \\ 0 & b & 0 & b \\ 0 & b & 0 & 0 \end{bmatrix} \div b \rightarrow \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & b & 0 \\ 0 & b & 0 & b \\ 0 & b & 0 & 0 \end{bmatrix} - b\,\mathrm{I} \rightarrow$$

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & b & 0 & b \\ 0 & b & 0 & 0 \end{bmatrix} \div b \rightarrow \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & b & 0 & 0 \end{bmatrix} - b\,\mathrm{III} \rightarrow$$

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & -b \end{bmatrix} \div (-b) \rightarrow \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} - \mathrm{IV}$$

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow[\text{rows}]{\text{Rearrange}} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

basis of image: $\begin{bmatrix} a \\ a \\ b \\ b \end{bmatrix}, \begin{bmatrix} b \\ b \\ a \\ a \end{bmatrix}, \begin{bmatrix} a \\ 0 \\ b \\ 0 \end{bmatrix}.$

② If $b = 0$: $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $\dim \operatorname{im}(A) = 0$, no basis.

To sum up

If $a \neq 0$, $b \neq a$, then basis of image is

$$\begin{bmatrix} a \\ a \\ a \\ a \end{bmatrix}, \quad \begin{bmatrix} a \\ a \\ b \\ b \end{bmatrix}, \quad \begin{bmatrix} a \\ 0 \\ b \\ 0 \end{bmatrix};$$

If $a \neq 0$, $b = a$, then basis of image is

$$\begin{bmatrix} a \\ a \\ a \\ a \end{bmatrix}, \quad \begin{bmatrix} a \\ 0 \\ b \\ 0 \end{bmatrix};$$

If $a = 0$, $b \neq 0$, then basis of image is

$$\begin{bmatrix} a \\ a \\ b \\ b \end{bmatrix}, \quad \begin{bmatrix} b \\ b \\ a \\ a \end{bmatrix}, \quad \begin{bmatrix} a \\ 0 \\ b \\ 0 \end{bmatrix};$$

If $a = 0$, $b = 0$, then $\operatorname{im}(A) = \{\vec{0}\}$, it has no basis.