

## REDUCED ROW ECHELON FORM AND GAUSS-JORDAN ELIMINATION

### 1. MATRICES

A *matrix* is a table of numbers. Ex:

$$\begin{bmatrix} 2 & 0 \\ 1 & -1 \\ 0 & 3 \end{bmatrix} \text{ or } \begin{bmatrix} 0 & -2 \\ 1 & 1 \end{bmatrix}.$$

A vertical line of numbers is called a *column* and a horizontal line is a *row*. A  $n \times m$  matrix has  $n$  rows and  $m$  columns. For instance, a general  $2 \times 4$  matrix,  $A$ , is of the form:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{bmatrix}.$$

Here the entry  $a_{ij}$  is the entry in the  $i$ th row and  $j$ th column.

Any linear system with  $n$  equations and  $m$  unknowns may be encoded as a  $n \times (m + 1)$  matrix called the *augmented matrix*. We illustrate this by example:

$$\begin{cases} y + z = 0 \\ x + y = 2 \\ x - y - z = 0 \end{cases} \longleftrightarrow \left[ \begin{array}{ccc|c} 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 2 \\ 1 & -1 & -1 & 0 \end{array} \right].$$

Collectively, the first  $m$  columns correspond to the coefficients of the unknowns in the system and taken together form a  $n \times m$  matrix called the *coefficient matrix*. The final column corresponds to the constant terms in the system and so we sometimes call it the *constant column*.

### 2. REDUCED ROW ECHELON FORM

Linear systems that are in a certain special form are extremely easy to solve. Rather than describe this form directly, we will define a special class of matrices. The linear systems whose augmented matrices are of this special class will be precisely those that are easy to solve.

We say an  $n \times m$  matrix  $A$  is in *reduced row echelon form* (*rref*) if the following are true of  $A$ :

- (1) Each non-zero row has first non-zero entry equal to 1 (called *leading 1* or *pivot*).
- (2) If a column contains a pivot, then every other entry in the column is zero.
- (3) As one moves down the rows, the pivots move strictly to the right. All zero rows (the only rows not containing a pivot) are at the bottom.

Notice (3) is a little more precise than what I stated in class as there was some ambiguity about where one was supposed to put zero rows.

Examples:

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Non-Examples:

$$\begin{bmatrix} 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 3 \\ 1 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Given an augmented matrix of a linear system in rref, we have the following rules for finding solutions to the corresponding system

- (1) If a leading 1 exists in the last column (i.e., the constant column), the system is inconsistent ( $0 = 1$ ).
- (2) Each column in the coefficient matrix *without* a pivot is a *free variable*, each column with a pivot is a *pivot variable*.
- (3) If system is not inconsistent, express pivot variables in terms of free variables and constants

Example: For a system with unknowns  $x, y, z$  and augmented matrix

$$\left[ \begin{array}{ccc|c} 1 & -2 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{array} \right]$$

the pivot variables are  $x, z$  and the only free variable is  $y$ . Solutions are of the form  $(1 + 2y, y, -2)$  where  $y$  is arbitrary

### 3. GAUSS-JORDAN ELIMINATION

Certain operations applied to an augmented matrix,  $A$ , of a given linear system will yield a new matrix,  $A'$ , which has the same solutions as the original system (more precisely  $A'$  will be the augmented matrix of a new system with the same solutions as the original one). These are called *elementary row operations*.

An elementary row operation applied to a matrix  $A$  is one of the following:

- (1) Swap two rows of  $A$  (not columns!).
- (2) Multiply a row of  $A$  by a non-zero number (zero not allowed!).
- (3) Add a multiple of any row of  $A$  to another row (zero allowed, but pointless!).

*Gauss-Jordan elimination* (or *Gaussian elimination*) is an algorithm which consists of repeatedly applying elementary row operations to a matrix so that after finitely many steps it is in rref. This is particularly useful when applied to the augmented matrix of a linear system as it gives a systematic method of solution.

The algorithm for a matrix  $A$  is:

- (1) Start with the first row of  $A$ .
- (2) Scale: If the row is non-zero, then multiply it by an appropriate scalar to obtain a pivot; otherwise, go to step (4).
- (3) Clear: Add appropriate multiples of the row to every other row in order to make the pivot the only non-zero entry in its column.
- (4) Start again at step (2) with the next row down.
- (5) Swap: Swap the rows so matrix in rref.

Given a matrix  $A$  this algorithm always terminates after finitely many steps and the output is a matrix in rref. Hence, it defines a function from the set of matrices to the set of matrices in rref. Denote by  $rref(A)$  the matrix obtained from  $A$  by Gauss-Jordan elimination.

It is true (EXERCISE: prove this) that the only matrix in rref that is obtainable from  $A$  by repeated application of elementary row operations is  $rref(A)$ . In other

words the algorithm gives just one path to  $rref(A)$ . This means, for instance, that you don't necessarily have to scale before clearing, but it is good practice to do so.

Example:

$$rref \begin{bmatrix} 2 & 6 & 0 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -1 \end{bmatrix}.$$

Indeed,

$$\begin{aligned} \begin{bmatrix} 2 & 6 & 0 \\ 1 & 2 & 1 \end{bmatrix} &\rightarrow \begin{bmatrix} 1 & 3 & 0 \\ 1 & 2 & 1 \end{bmatrix} && \text{scale first row to obtain pivot} \\ &\rightarrow \begin{bmatrix} 1 & 3 & 0 \\ 0 & -1 & 1 \end{bmatrix} && \text{clear first column using pivot} \\ &\rightarrow \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & -1 \end{bmatrix} && \text{scale second row to obtain pivot} \\ &\rightarrow \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -1 \end{bmatrix} && \text{clear second column using pivot, in rref so no need to swap.} \end{aligned}$$

#### 4. NUMBER OF SOLUTIONS

Let us now briefly discuss what we can say about the number of solutions of a linear system using Gauss-Jordan elimination. The basic observation is that for a linear system with augmented matrix  $A$  we have following relationships between the number of solutions of the system and  $rref(A)$ .

- (1) If there is a pivot in the final column of  $A$  (i.e., in the constant column), then the system is inconsistent, i.e., there is no solution. The reverse is also true, if there is no solution, then there must be a pivot in the final column.
- (2) If there is no pivot in the final column and every other column contains a pivot (i.e., every column of the coefficient matrix contains a pivot), then there is a unique solution. The reverse is also true, if there is a unique solution, then every column but the constant column must contain a pivot.
- (3) If there is no pivot in the final column and at least one other column does not contain a pivot (i.e., one column of coefficient matrix does not contain a pivot), then there are an infinite number of solutions. The reverse is also true, if there are an infinite number of solutions, then the constant column and at least one column of the coefficient matrix do not contain pivots.

This is straightforward to check based on our discussion at the end of Section 2. A consequence of this is the following theorem we discussed in class

**Theorem 4.1.** *Consider a system of linear equations, then the following are true:*

- a) *If the system has a unique solution, then it has at least as many equations as unknowns.*
- b) *If the system has strictly fewer equations than unknowns it has either no solution or an infinite number of solutions.*

*Proof.* First prove statement a): Let  $A$  be the augmented matrix of the system. If this matrix is  $n \times (m + 1)$ , then that means the system has  $n$  equations and  $m$  unknowns. As the system has a unique solution,  $rref(A)$  cannot have a pivot in the final column and also cannot have any other columns without a pivot. That is, there are  $m$  pivots. As each pivot sits in its own row, there must be at least  $m$  rows, that is  $n \geq m$  as claimed. Statement b) is the contrapositive of a) so follows immediately.  $\square$

## 5. MATRIX RANK

We define the *rank* of a matrix  $A$  to be the number of pivots in  $rref(A)$ . We will denote the rank of  $A$  by  $rank(A)$ .

Example:

$$rank \begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = 2.$$

This is because

$$rref \begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \text{ has two pivots.}$$

For a linear system, the rank of its augmented matrix or of its coefficient matrix can be used to give conditions on the number of solutions to the system. A useful fact in this regard (convince yourself of why this is true): If  $C$  is the coefficient matrix of an augmented matrix  $A$  of some linear system, then  $rref(C)$  is the coefficient matrix of  $rref(A)$ .

Example: If a linear system has coefficient matrix  $C$  which is  $n \times m$  and  $rank(C) = n$ , then the system has at least one solution and if  $n = m = rank(C)$ , then there is a unique solution. To see why this is true observe that if  $A$  is the augmented matrix of the system, then all the pivots of  $rref(A)$  are in the coefficient matrix (if there was one in the constant column it would require an extra row to hold it) and so the system can't be inconsistent. If, in addition  $m = n$ , then every column of the coefficient matrix contains a pivot so we are in case (2) of Section 4.