# SPECTRAL THEOREM

Orthogonal Diagonalizable A diagonal matrix $D$ has eigenbasis $\mathcal{E} = (\vec{e}_1, \ldots, \vec{e}_n)$ which is an orthonormal basis. It's a natural question to ask when a matrix $A$ can have an orthonormal basis. As such we say, $A \in \mathbb{R}^{n\times n}$ is *orthogonally diagonalizable* if $A$ has an eigenbasis $\mathcal{B}$ that is also an orthonormal basis. This is equivalent to the statement that there is an orthogonal matrix $Q$ so that $Q^{-1}AQ = Q^\top AQ = D$ is diagonal.

**Theorem 0.1.** *If $A$ is orthogonally diagonalizable, then $A$ is symmetric.*

*Proof.* By definition, there is an orthogonal matrix $Q$ so that

$$Q^{-1}AQ = Q^\top AQ = D \Rightarrow A = QDQ^{-1}$$

where $D$ is diagonal. As $D^\top = D$, we have

$$D = D^\top = (Q^\top AQ)^\top = Q^\top A^\top (Q^\top)^\top = Q^{-1}A^\top Q.$$

Here we used that orthogonal matrices satisfy $Q^\top = Q^{-1}$. Hence,

$$A^\top = QDQ^{-1} = A$$

$\square$

EXAMPLE: $A = \begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix}$ is not orthogonally diagonalizable as $A^\top \neq A$.

Remarkably, the converse to this theorem is also true.

**Theorem 0.2.** *(Spectral theorem) $A \in \mathbb{R}^{n\times n}$ is orthogonally diagonalizable if and only if it is symmetric.*

An important consequence of this is that a symmetric $n \times n$ matrix has (counting multiplicities) exactly $n$ (real) eigenvalues

EXAMPLE: Orthogonally diagonalize

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

This is symmetric so can be orthogonally diagonalized by the spectral theorem. By inspection, $\operatorname{rank}(A) = 1$ so $\operatorname{null}(A) = 2$. This means that 0 is an eigenvalue of $A$ with geometric multiplicity 2. Similarly, by inspection, $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is an eigenvector with eigenvalue 3. As $3 \geq \operatorname{gemu}(0) + \operatorname{gemu}(3) \geq 2 + 1 = 3$ we can conclude that we have found all possible eigenvalues. An elementary computation implies,

$$E_0 = \ker(A) = \operatorname{span}\left( \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right)$$

and

$$E_3 = \operatorname{span}\left(\begin{bmatrix}1\\1\\1\end{bmatrix}\right).$$

Notice, $E_3 = E_0^{\perp}$ (recall this holds in general–see below). Hence, to find a orthonormal eigenbasis, enough to find an orthonormal eigenbasis of $E_0$ and of $E_3$ and then concatenate the two. Applying Gram-Schmidt to the basis of $E_0$ we found, we obtain

$$E_0 = \operatorname{span}\left(\begin{bmatrix}-\frac{\sqrt{2}}{2}\\ \frac{\sqrt{2}}{2}\\ 0\end{bmatrix}, \begin{bmatrix}-\frac{\sqrt{6}}{6}\\ -\frac{\sqrt{6}}{6}\\ \frac{\sqrt{6}}{3}\end{bmatrix}\right)$$

and

$$E_3 = \operatorname{span}\left(\begin{bmatrix}\frac{\sqrt{3}}{3}\\ \frac{\sqrt{3}}{3}\\ \frac{\sqrt{3}}{3}\end{bmatrix}\right).$$

Hence, if we set

$$Q = \begin{bmatrix}-\frac{\sqrt{2}}{2} & -\frac{\sqrt{6}}{6} & \frac{\sqrt{3}}{3}\\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{6}}{6} & \frac{\sqrt{3}}{3}\\ 0 & \frac{\sqrt{6}}{3} & \frac{\sqrt{3}}{3}\end{bmatrix}.$$

then $Q$ is orthogonal and

$$Q^{-1}AQ = \begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 3\end{bmatrix}.$$

## 1. Sketch of Proof of Spectral Theorem

In order to prove the spectral theorem, we will need the following weaker statement:

**Theorem 1.1.** *If $A \in \mathbb{R}^{n\times n}$ is symmetric, then it has exactly $n$ eigenvalues counting multiplicities.*

*Proof.* This theorem is most easily proved using complex numbers (see textbook for details). It can also be proved using ideas from calculus. We will prove it for $2 \times 2$. In this case, $A$ being symmetric means

$$A = \begin{bmatrix}a & b\\ b & d\end{bmatrix},$$

we have

$$f_A(\lambda) = \lambda^2 - (a+d)\lambda + (ad - b^2)$$

The discriminant of $f_A$ is $D = (a+d)^2 - 4(ad-b^2) = a^2 + 2ad + d^2 - 4ad + 4b^2 = (a-d)^2 + 4b^2 \geq 0$ Hence, $f_A$ always has real roots. It has distinct real roots unless $a = d$ and $b = 0$, i.e., unless $A$ is diagonal. □

This says that we have enough eigenvalues to diagonalize, we just need to find an eigenbasis of $A$ that is an orthonormal basis. If all eigenvalues are distinct, this is not hard, as eigenvectors with different eigenvalues are orthogonal so we just take eigenbasis consisting of unit vectors. To see this claim observe that

$$\lambda_i \vec{v}_i \cdot \vec{v}_j = A\vec{v}_i \cdot \vec{v}_j = \vec{v}_i \cdot A^{\top}\vec{v}_j = \vec{v}_i \cdot A\vec{v}_j = \lambda_j \vec{v}_i \cdot \vec{v}_j.$$

Thus, if $\lambda_i \neq \lambda_j$, then $\vec{v}_i \cdot \vec{v}_j = 0$.

The situation is more complicated if there is repeated eigenvalues. For instance, one might worry the matrix is "defective," that is the sum of the geometric multiplicities might be less than $n$. When $n = 2$ we already saw the matrix is diagonal so trivial in this case and can show this doesn't happen for larger $n$. Arguing as in the example above, one finds the orthonormal eigenbasis by applying Gram-Schmidt to each eigenspace and then concatenating.