LINEAR SUBSPACES

1. Subspaces of $\mathbb{R}^n$

We wish to generalize the notion of lines and planes. To that end, say a subset $W \subset \mathbb{R}^n$ is a *(linear) subspace* if it has the following three properties:

(1) (Non-empty): $\vec{0} \in W$;
(2) (Closed under addition): $\vec{v}_1, \vec{v}_2 \in W \Rightarrow \vec{v}_1 + \vec{v}_2 \in W$;
(3) (Closed under scaling): $\vec{v} \in W$ and $k \in \mathbb{R} \Rightarrow k\vec{v} \in W$.

EXAMPLE: $\left\{\vec{0}\right\}$ and $\mathbb{R}^n$ are subspaces of $\mathbb{R}^n$ for any $n \geq 1$.

EXAMPLE: Line $\left\{t \begin{bmatrix} 1 \\ 1 \end{bmatrix} : t \in \mathbb{R}\right\}$ given by $x_1 = x_2$ is a subspace.

NON-EXAMPLE: $W = \left\{\begin{bmatrix} x \\ y \end{bmatrix} : x \geq 0, y \geq 0\right\}$. As $\begin{bmatrix} 1 \\ 1 \end{bmatrix} \in W$, but $-\begin{bmatrix} 1 \\ 1 \end{bmatrix} \notin W$.

EXAMPLE: If $T : \mathbb{R}^m \to \mathbb{R}^n$, then $\ker(T)$ is a subspace and $\mathbb{R}^m$ and $\mathrm{Im}\,(T)$ is a subspace of $\mathbb{R}^n$. For instance,

(1) $T(\vec{0}) = \vec{0} \Rightarrow \vec{0} \in \ker(T)$;
(2) $\vec{v}_1, \vec{v}_2 \in \ker(T) \Rightarrow T(\vec{v}_1 + \vec{v}_2) = T(\vec{v}_1) + T(\vec{v}_2) = \vec{0} \Rightarrow \vec{v}_1 + \vec{v}_2 \in \ker(T)$;
(3) $\vec{v} \in \ker(T), k \in \mathbb{R} \to T(k\vec{v}) = kT(\vec{v}) = k\vec{0} = \vec{0} \Rightarrow k\vec{v} \in \ker(T)$.

2. Span of vectors

Given a set of vectors $\vec{v}_1, \ldots, \vec{v}_m \in \mathbb{R}^n$ we define the *span* of these vectors to be

$$W = span(\vec{v}_1, \ldots, \vec{v}_k) = \{c_1\vec{v}_1 + \ldots + c_k\vec{v}_k : c_1, \ldots, c_m \in \mathbb{R}\}$$

In other words, $\vec{w} \in W$ means $\vec{w}$ is a linear combination of the $\vec{v}_i$. If

$$A = \begin{bmatrix} \vec{v}_1 & | & \cdots & | & \vec{v}_m \end{bmatrix}$$

is the $n \times m$ matrix with columns $\vec{v}_i$, then

$$\mathrm{span}(\vec{v}_1, \ldots, \vec{v}_k) = \mathrm{Im}\,(A)$$

and so $\mathrm{span}(\vec{v}_1, \ldots, \vec{v}_k)$ is a subspace of $\mathbb{R}^n$. Thought of the other way around, the image of A is the span of its columns.

Given a subspace $W \subset \mathbb{R}^n$ we say a set of vectors $\vec{v}_1, \ldots, \vec{v}_m \in \mathbb{R}^n$ *span* W if

$$W = \mathrm{span}(\vec{v}_1, \ldots, \vec{v}_m).$$

EXAMPLE: $\mathrm{span}(\vec{0}) = \left\{\vec{0}\right\}$.

EXAMPLE: If $\vec{e}_1, \vec{e}_2$ are standard vectors in $\mathbb{R}^2$, then $\mathrm{span}(\vec{e}_1, \vec{e}_2) = \mathbb{R}^2$.

EXAMPLE: $\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} \in \mathrm{span}\left(\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right)$ and $\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \notin \mathrm{span}\left(\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right)$.

$$\mathrm{rref}\begin{bmatrix} 1 & 0 & | & 2 \\ 0 & 1 & | & 3 \\ -1 & 1 & | & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & | & 2 \\ 0 & 1 & | & 3 \\ 0 & 0 & | & 0 \end{bmatrix}$$

so

$$\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + 3 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

Next compute

$$\operatorname{rref} \begin{bmatrix} 1 & 0 & | & 2 \\ 0 & 1 & | & 1 \\ -1 & 1 & | & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & | & 0 \\ 0 & 1 & | & 0 \\ 0 & 0 & | & 1 \end{bmatrix}$$

As this is inconsistent, and so there is no way to write:

$$\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

.

3. Linear independence

Many different sets of vectors may span the same subspace. For instance,

$$W = \operatorname{span}(\vec{v}_1, \vec{v}_2, \vec{v}_1 + \vec{v}_2) = \operatorname{span}(\vec{v}_1, \vec{v}_2).$$

Indeed, $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3(\vec{v}_1 + \vec{v}_2) = (c_1 + c_3)\vec{v}_1 + (c_1 + c_3)\vec{v}_2$. In other words, $\vec{v}_1 + \vec{v}_2$ is *redundant* and is not needed to describe the subspace W. To formalize this, define a *linear relation* among $\vec{v}_1, \dots, \vec{v}_m$ to be an equation of the form

$$c_1\vec{v}_1 + \cdots + c_m\vec{v}_m = 0.$$

There is always such a solution with $c_1 = \cdots c_m = 0$. A *non-trivial relation* is one in which at least one $c_i \neq 0$

EXAMPLE:

$$2\begin{bmatrix} 1 \\ 1 \end{bmatrix} - 3\begin{bmatrix} 1 \\ 0 \end{bmatrix} - \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \vec{0}$$

is a non-trivial linear relation amongst three vectors and this can be rewritten

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{2}{3}\begin{bmatrix} 1 \\ 1 \end{bmatrix} - \frac{1}{3}\begin{bmatrix} -1 \\ 2 \end{bmatrix}.$$

EXAMPLE: Suppose $\vec{v}_1, \dots, \vec{v}_m$ admit a non-trivial relation

$$c_1\vec{v}_1 + \cdots + c_m\vec{v}_m = 0$$

with $c_m \neq 0$, then $\operatorname{span}(\vec{v}_1, \cdots, \vec{v}_{m-1}) = \operatorname{span}(\vec{v}_1, \cdots, \vec{v}_m)$. That is the vector $\vec{v}_m$ is redundant and can be omitted.

If

$$A = \begin{bmatrix} \vec{v}_1 & | & \cdots & | & \vec{v}_m \end{bmatrix}$$

is the matrix with columns the vectors $\vec{v}_i$, then a linear relation can be identified with a unique element of $\ker(A)$. Indeed,

$$c_1\vec{v}_1 + \ldots + c_m\vec{v}_m = \vec{0} \iff \begin{bmatrix} c_1 \\ \vdots \\ c_m \end{bmatrix} \in \ker(A).$$

Furthermore, a non-trivial relation corresponds to a non-zero entry of $\ker(A)$.

We say $\vec{v}_1, \dots, \vec{v}_m$ are *linearly independent* if they have no non-trivial relation, that is, if $ker(A) = \left\{\vec{0}\right\}$. Put another way, $\ker(A) = \left\{\vec{0}\right\}$ if and only if the columns of A are linearly independent.

EXAMPLE: If $\vec{v}_1, \ldots, \vec{v}_m$ $(m \geq 2)$ are linearly independent, then

$$\operatorname{span}(\vec{v}_1, \ldots, \vec{v}_{m-1}) \subsetneq \operatorname{span}(\vec{v}_1, \ldots, \vec{v}_m).$$

That is, there is no redundancy for linearly independent sets of vectors. More generally none of the vectors can be omitted without making the span smaller.

EXAMPLE: $(m = 1)$ $\vec{v}_1$ is linear independent if and only if $\vec{v}_1 \neq 0$.

EXAMPLE: $\vec{v}_1, \vec{v}_2, \vec{v}_1 + \vec{v}_2$ are never linearly independent ($c_1 = c_2 = 1, c_3 = -1$ gives a non-trivial relation).

EXAMPLE: $\vec{e}_1, \vec{e}_2$ are linearly independent in $\mathbb{R}^2$. Indeed, if

$$c_1\vec{e}_1 + c_2\vec{e}_2 = \vec{0} \iff \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

so $c_1 = c_2 = 0$. That is, the only linear relation is the trivial one.

EXAMPLE: Suppose $\operatorname{span}(\vec{v}_1, \vec{v}_2, \vec{v}_3) = \operatorname{span}(\vec{v}_1 + \vec{v}_2, \vec{v}_1 - \vec{v}_2)$, then $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are not linearly independent. Indeed, $\vec{v}_3 = c_1(\vec{v}_1 + \vec{v}_2) + c_2(\vec{v}_1 - \vec{v}_2)$ which gives the non-trivial linear relation

$$-(c_1 + c_2)\vec{v}_1 - (c_1 - c_2)\vec{v}_2 + \vec{v}_3 = \vec{0}.$$

EXAMPLE: Are $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$ linearly independent? First compute

$$\operatorname{rref} \begin{bmatrix} 1 & -2 & -1 \\ 1 & -1 & 1 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}.$$

This means that the kernel of this matrix is

$$\operatorname{span}\left(\begin{bmatrix} -3 \\ -2 \\ 1 \end{bmatrix}\right)$$

That is, there is have non-trivial linear relation

$$-3\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - 2\begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} = \vec{0}$$

and so the vectors are not linearly independent.

EXAMPLE: Are $\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ linearly independent? Compute

$$\ker \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} = \left\{\vec{0}\right\}$$

by showing that

$$\operatorname{rref} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} = I_3.$$

4. Basis of a subspace

A set of vectors which span a subspace W and which does not have any redundancies is clearly of particular interest. With this in mind for a subspace $W \subset \mathbb{R}^n$, we say a set $\{\vec{v}_1, \ldots, \vec{v}_m\}$ is a *basis of* W if

(1) $\vec{v}_1, \ldots, \vec{v}_m$ are linearly independent
(2) $W = \operatorname{span}(\vec{v}_1, \ldots, \vec{v}_m)$.

This ensures that not only do the vectors span the subspace but none of them can be omitted.

EXAMPLE: $\vec{e}_1, \vec{e}_2$ is a basis of $\mathbb{R}^2$. Indeed, any vector

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1\vec{e}_1 + \vec{e}_2$$

so the vectors span. They are linearly indepdent by inspection.

NON-EXAMPLE: $\vec{e}_1 + \vec{e}_2$ is not a basis of $\mathbb{R}^2$ as it does not span (but is linearly independent).

NON-EXAMPLE: $\vec{e}_1, \vec{e}_2, \vec{e}_1 + \vec{e}_2$ is not a basis of $\mathbb{R}^2$ as the set is not linearly independent (but does span).

EXAMPLE: The subspace $\left\{\vec{0}\right\}$ has no basis. Say its basis is the empty set, $\emptyset$.

EXAMPLE: Find basis of $\ker \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 1 & 0 \end{bmatrix}$. Observe this matrix is already in RREF so it is easy to determine elements in the kernel. Indeed, there are two free variables $f_1 = x_3$ and $f_2 = x_4$. Plugging in specific values for these variables (i.e., solving for the pivot variables) gives the following two elements in the kernel:

$$\vec{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} (f_1 = x_3 = 1, f_2 = x_4 = 0) \text{ and } \vec{v}_2 = \begin{bmatrix} -2 \\ 0 \\ 0 \\ 1 \end{bmatrix} (f_1 = x_3 = 0, f_2 = x_4 = 1)$$

Clearly, if $f_1 = s$ and $f_2 = t$, then a general element of the kernel is of the form $\vec{z} = s\vec{v}_1 + t\vec{v}_2$. In other words, the kernel is spanned by $\vec{v}_1$ and $\vec{v}_2$. Hence, we just need to check that $\vec{v}_1, \vec{v}_2$ are linearly independent to see they are a basis. However,

$$\vec{0} = c_1\vec{v}_1 + c_2\vec{v}_2 = \begin{bmatrix} c_1 - 2c_2 \\ -c_1 \\ c_1 \\ c_2 \end{bmatrix}.$$

Hence, we must have $c_1 = 0$ (from the third entry) and $c_2 = 0$ (from the fourth entry). That is the only linear relation is the trivial one and so the vectors are linearly independent and so form a basis of the kernel.

This example can be generalized to give a procedure for finding the basis of $\ker(A)$ for any matrix A. A basic observation is that $\ker(A) = \ker(\operatorname{rref}(A))$. This is becaus the linear system with augmented matrix $[A|\vec{0}]$ has the same solutions as that of $\operatorname{rref}[A|\vec{0}] = [\operatorname{rref}(A)|\vec{0}]$. With that in mind:

(1) Compute $\operatorname{rref}(A)$ and use $\operatorname{rref}(A)$ to determine free variables.
(2) Label the free variables as $f_1, \ldots, f_p$.
(3) Let $\vec{v}_i$ be solutions corresponding to $f_i = 1$ and $f_j = 0$ for $j \neq i$. That is, use $\operatorname{rref}(A)$ to solve for the pivot variables in terms of the specified values of the free variables.

(4) Basis of $\ker(A)$ is $\vec{v}_1, \ldots, \vec{v}_p$.

In order to justify this first observe that $\ker(A) = \mathrm{span}(\vec{v}_1, \ldots, \vec{v}_p)$. As the element of $\ker(A)$ corresponding to $f_1 = t_1, \ldots, f_p = t_p$ is $t_1\vec{v}_1 + \ldots + t_p\vec{v}_p \in \ker(A)$. Similarly, only $\vec{v}_j$ has a non-zero entry at row corresponding to f_j, so easy to see $\vec{v}_1, \ldots, \vec{v}_p$ are linearly independent and hence form a basis.

EXAMPLE: Find a basis of $\mathrm{Im}\,(A)$ for

$$A = \begin{bmatrix} 1 & 3 & -1 \\ 0 & -2 & 1 \\ 2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \vec{w}_1 & | & \vec{w}_2 & | & \vec{w}_3 \end{bmatrix}.$$

We compute

$$\mathrm{rref}(A) = \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix}$$

Hence,

$$\ker(A) = \mathrm{span}\left(\begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right) = \mathrm{span}\left(\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}\right)$$

This means that there is a linear relation amongst the columns of A:

$$-\frac{1}{2}\vec{w}_1 + \frac{1}{2}\vec{w}_2 + \vec{w}_3 = -\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + \begin{bmatrix} 3 \\ -2 \\ 0 \end{bmatrix} + 2\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} = \vec{0} \iff \vec{w}_3 = \frac{1}{2}\vec{w}_1 - \frac{1}{2}\vec{w}_2.$$

This shows that $\vec{w}_3 \in \mathrm{span}(\vec{w}_1, \vec{w}_2)$. Furthermore, any linear relation

$$\vec{0} = c_1\vec{w}_1 + c_2\vec{w}_2 = \begin{bmatrix} c_1 + 3c_2 \\ -2c_2 \\ 2c_1 \end{bmatrix}$$

which implies $0 = 2c_1 = -2c_2$ and so $\vec{w}_1, \vec{w}_2$ are linearly independent and hence $\vec{w}_1, \vec{w}_2$ form a basis of $\mathrm{Im}\,(A)$.

This example can also be generalized to give a basis of $\mathrm{Im}\,(A)$ when

$$A = \begin{bmatrix} \vec{w}_1 & | & \cdots & | & \vec{w}_n \end{bmatrix}.$$

It is worth noting that, in general, $\mathrm{Im}\,(A) \neq \mathrm{Im}\,(\mathrm{rref}(A))$. Nevertheless, $\mathrm{rref}(A)$ can be used to find the basis. The procedure is as follows:

(1) Compute $\mathrm{rref}(A)$ and use this to find the pivot variables.
(2) Let $\vec{y}_1 = \vec{w}_{i_1}, \ldots, \vec{y}_q = \vec{w}_{i_q}$ be the columns of A that correspond to the pivot variables. Call these the *pivot columns of* A. That is, a pivot column of A is a one which corresponds to a column of $\mathrm{rref}(A)$ that contains a pivot.
(3) A basis of $\mathrm{Im}\,(A)$ is $\vec{y}_1, \ldots, \vec{v}_q$. That is, the pivot columns of A are a basis of $\mathrm{Im}\,(A)$.

To understand why this is the case, we will use the vectors $\vec{v}_i$ from before. Indeed,

$$A\vec{v}_i = \vec{0}$$

corresponds to a non-trivial linear relation between the non-pivot column corresponding to the ith free variable and all of the pivot columns. In other words, this non-pivot column lies in $\mathrm{span}(\vec{y}_1, \ldots, \vec{y}_q)$. As this holds for each non-pivot column,

$$\mathrm{Im}\,(A) = \mathrm{span}(\vec{y}_1, \ldots, \vec{y}_q).$$

To see why the $\vec{y}_1, \dots, \vec{y}_q$ are linearly independent, we observe that any non-zero element of $\ker(A)$ must have a non-zero entry in one of the rows corresponding to free variable. This is because otherwise each free variable is 0 and so the corresponding element of $\ker(A)$ is $\vec{0}$. As any linear relation among $\vec{y}_1, \dots, \vec{y}_q$ can be thought of as an element of $\ker(A)$ whose entries in the rows corresponding to the free variables are 0, we see that there are no non-trivial relation among the pivot columns. That is, the pivot columns are linearly independent and so form a basis.

5. Dimension of subspaces

Fix a subspace $W \subset \mathbb{R}^n$. We pose two natural questions:

(1) How many vectors are needed to span W?
(2) How many vectors in W can be linearly independent?

EXAMPLE: When $W = \mathbb{R}^n$, then need at least n vectors to span. Indeed,

$$\mathbb{R}^n = \text{span}(\vec{v}_1, \dots, \vec{v}_p) \iff \text{Im}\begin{bmatrix}\vec{v}_1 & | & \cdots & | & \vec{v}_p\end{bmatrix} = \mathbb{R}^n$$
$$\iff \text{rank}\begin{bmatrix}\vec{v}_1 & | & \cdots & | & \vec{v}_p\end{bmatrix} = n \Rightarrow p \geq n.$$

EXAMPLE: When $W = \mathbb{R}^n$. If $\vec{w}_1, \dots, \vec{w}_q \in \mathbb{R}^n$ are linearly independent, then $q \leq n$. Indeed, vectors linearly independent means

$$\ker\begin{bmatrix}\vec{w}_1 & | & \cdots & | & \vec{w}_q\end{bmatrix} = \left\{\vec{0}\right\} \iff \text{rank}\begin{bmatrix}\vec{w}_1 & | & \cdots & | & \vec{w}_q\end{bmatrix} = q \Rightarrow q \leq n.$$

Theorem 5.1. *Fix a subspace $W \subset \mathbb{R}^n$. If $W = span(\vec{v}_1, \dots, \vec{v}_p)$ and $\vec{w}_1, \dots, \vec{w}_q \in W$ are linearly independent, then $p \geq q$.*

Proof. Let

$$A = \begin{bmatrix}\vec{v}_1 & | & \cdots & | & \vec{v}_p\end{bmatrix} \text{ and } B = \begin{bmatrix}\vec{w}_1 & | & \cdots & | & \vec{w}_q\end{bmatrix}.$$

A is $n \times p$ and B is $n \times q$ Our hypotheses ensure $\text{Im}\,(A) = W$ and $\ker(B) = \left\{\vec{0}\right\}$. Clearly, $\vec{w}_i \in W = \text{Im}\,(A)$ so there are $y_i \in \mathbb{R}^p$ so that $\vec{w}_i = A\vec{y}_i$. Let

$$C = \begin{bmatrix}\vec{y}_1 & | & \cdots & | & \vec{y}_q\end{bmatrix}$$

be $p \times q$. We have $B = AC$. As $\ker(B) = \left\{\vec{0}\right\}$ and $\ker(B) = ker(AC) \supset \ker(C)$, and so $\ker(C) = \left\{\vec{0}\right\}$. This means $\text{rank}(C) = q$ and so $p \geq q$ as claimed. □

Corollary. *If $\vec{v}_1, \dots, \vec{v}_p$ and $\vec{w}_1, \dots, \vec{w}_q$ are both a basis of W, then $p = q$.*

Proof. $\vec{v}_1, \dots, \vec{v}_p$ is linearly independent (spans) and $\vec{w}_1, \dots, \vec{w}_q$ spans (is linearly independent), so $p \leq q$ ($q \leq p$). Both inequalities are true so $p = q$. □

This means there is a well-defined notion of dimension of a subspace. Specifically, iff $W \subset \mathbb{R}^n$ is a subspace, then the *dimension*, $\dim(W)$, of W is the number of elements in a basis of W. The corollary ensures this number is does not depend on the choice of basis. Strictly speaking, for this definition to make sense for *every* subspace need to know it has a basis. You did this in your homework.

Using the Theorem we just proved we make the following observations for $W \subset \mathbb{R}^n$ a subspace with $dim(W) = m$:

(1) One can find at most m linearly independent vectors in W.
(2) Spanning W requires at least m vectors.
(3) If m vectors in W are linearly independent, then they are a basis of W.

(4) If m vectors span W, then they are a basis of W.

EXAMPLE: $\dim(\left\{\vec{0}\right\}) = 0$ because $\left\{\vec{0}\right\}$ has basis the empty set.
EXAMPLE: For $\vec{v} \neq 0$, $\dim(\mathrm{span}(\vec{v})) = 1$, i.e., a line has dimension 1.
EXAMPLE: $\dim(\mathbb{R}^n) = n$ because $\mathbb{R}^n$ has the *standard basis*, $\vec{e}_1, \ldots, \vec{e}_n$.
EXAMPLE: $\dim(\mathrm{Im}\,(A)) = \mathrm{rank}(A)$, as pivot columns of A are a basis of $\mathrm{Im}\,(A)$.
EXAMPLE: If $W \subset V \subset \mathbb{R}^n$, then $\dim(W) \leq \dim(V)$.
EXAMPLE: $W \subset \mathbb{R}^n$ a subspace, then $\dim(W) \leq n$.

6. Rank-Nullity Theorem

We can relate the dimension of $\ker(A)$ and $\mathrm{Im}\,(A)$ to the number of columns of A. This theorem is sometimes called the *fundamental theorem of linear algebra* due to its importance. With this in mind, call $\dim(\ker(A)$ the *nullity* of A and write $\mathrm{null}(A) := \dim(\ker(A))$.

Theorem 6.1. *Let A be a $n \times m$ matrix, then,*

$$\dim(\ker(A)) + \dim(\mathrm{Im}\,(A)) = \mathrm{null}(A) + \mathrm{rank}(A) = m.$$

Proof. As mentioned, $\mathrm{rank}(A)$ is the number of pivot columns. Likewise, $\mathrm{null}(A)$ is the number of non-pivot columns. This is because, each non-pivot column corresponds to a unique element of the basis of $ker(A)$ constructed earlier. As each column of A is either a pivot column or a non-pivot column, the result follows. □

EXAMPLE: Can a 3×3 matrix, A, have $\ker(A) = \mathrm{Im}\,(A)$? The answer is no, as that would mean $\mathrm{rank}(A) = \mathrm{null}(A)$, but $3 = \mathrm{rank}(A) + \mathrm{null}(A)$ is not even.
EXAMPLE: Let A be a $n \times p$ matrix and B be a $p \times m$ matrix we have

$$\mathrm{null}(AB) \geq \mathrm{null}(B).$$

This is because $\ker(B) \subset \ker(AB)$.