Math 201
Spring 2014
Final
05/07/14
Lecturer: Jesus Martinez Garcia
Time Limit: 3 hours

Name (Print): ____________________

Section: ____________________

This exam contains 13 pages (including this cover page) and 6 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- **If you use a theorem of lemma you must indicate this** and explain why the theorem may be applied.
- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- **Write with blue or black pen only**. We need to keep the exams for a year and pencil would fade out.
- **The last question is a bonus question**. You do not need to attempt it. It can only increase your grade.
- If you need more space, use the back of the pages; clearly indicate when you have done this.

Do not write in the table to the right.

Problem	Points	Score
1	20	
2	40	
3	15	
4	10	
5	15	
6	10	
Total:	110	

1. (20 points) **Inner product spaces.**

   (a) (5 points) Let V be a vector space. Give the definition of an inner product on V.

   (b) (5 points) Let $C^\infty(0,1)$ be the set of differentiable functions $f\colon (0,1) \to \mathbb{R}$. Show that

   $$\langle f, g\rangle = \int_0^1 \Big(f(x)g(x) + f'(x)g'(x)\Big)\,dx$$

   is an inner product on V.

(c) (10 points) Compute **all** the Fourier coefficients of $f(t) = t^2$.

2. (40 points) **Bases, Image, Kernel and dimension.** Please, note that the following exercises can be attempted independently. You may assume previous parts even if you have not completed them (e.g. you can assume part (a), T is linear, when solving part (d), even if you do not prove that T is linear).

   Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \mathrm{Mat}_2(\mathbb{R})$ and $T\colon \mathrm{Mat}_2(\mathbb{R}) \to \mathrm{Mat}_2(\mathbb{R})$ be defined by $T(M) = AM - MA$.

   (a) (5 points) Show that T is a linear transformation

   (b) (5 points) Show that

   $$\mathfrak{B} = \left\{ v_1 = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix},\ v_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ v_3 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix},\ v_4 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \right\}$$

   is a basis of $\mathrm{Mat}_2(\mathbb{R})$.

(c) (10 points) Let

$$\mathfrak{E} = \left\{ e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},\ e_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ e_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},\ e_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

be another basis of $\mathrm{Mat}_2(\mathbb{R})$. Compute the change of basis matrix $S = S_{\mathfrak{B}\to\mathfrak{E}}$

(d) (10 points) Find B, the $\mathfrak{B}$-matrix of T.

(e) (5 points) Give a basis of $\mathrm{Im}(T)$. What is the dimension of $\mathrm{Im}(T)$?

(f) (5 points) Give a basis of $\mathrm{Ker}(T)$. What is the dimension of $\mathrm{Ker}(T)$?

3. (15 points) **True/False.**

   Decide if the following statements are True or False. If they are true, provide a proof. If they are false, provide a counter-example.

   (a) (5 points) Let A be an invertible matrix. Then $(A^2)^{-1} = (A^{-1})^2$.

   (b) (5 points) Let $A, B \in \mathrm{Mat}_2(\mathbb{R})$. If $AB = 0$, then $BA = 0$.

   (c) (5 points) The space P_7 (polynomials of degree at most 7) is isomorphic to $\mathrm{Mat}_{2\times 4}(\mathbb{R})$.

4. (10 points) **Determinants and orthogonal matrices.**

   Let
   $$A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 3 \end{pmatrix}.$$

   Find A^{-1} by computing the adjoint of A.

5. (15 points) Let
$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}.$$
Let $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ given by $T(\overrightarrow{x}) = A\overrightarrow{x}$. Find an **orthonormal** basis $\mathfrak{B} = \{\overrightarrow{v}_1, \overrightarrow{v}_2\}$ such that $T(\overrightarrow{v}_1)$ and $T(\overrightarrow{v}_2)$ are orthogonal. What are the norms of $T(\overrightarrow{v}_1)$ and $T(\overrightarrow{v}_2)$?

6. (10 points) **BONUS QUESTION:** Find the matrix $A \in \mathrm{Mat}_n(\mathbb{R})$ of the orthogonal projection of $\mathbb{R}^n$ onto the line spanned by the vector $\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$.

   (Hint: Try first for $n = 3$).