These are sketches of solutions just to check that you got the answers right.

(1)
- The reduced row echelon form of the given matrix is
$$\operatorname{rref}(A) = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$
Thus the first three columns are pivot columns and as a basis one can take the first three columns.
- Using the reduced row echelon form above and solving for the pivot variables in terms of the free variables we see that the basis is given by the equations $x_1 = -x_4 - x_5$, $x_2 = -x_4$, $x_3 = -x_5$, so the kernel is given by
$$\left\{ \begin{pmatrix} -x_4 - x_5 \\ -x_4 \\ -x_5 \\ x_4 \\ x_5 \end{pmatrix} : x_4, x_5 \in \mathbb{R} \right\} = \operatorname{span}\left(\begin{pmatrix} -1 \\ -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} \right)$$
and the vectors in this span are linearly independent, therefore forming a basis.
- This can only work for $r = 1$ as all the columns have equal first and forth component. It indeed does work for $r = 1$ because the first column plus the third minus the fourth is equal to
$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

(2)
- First row: yes, yes, no. Second row: yes, yes, yes. Third row: yes, no, no. Fourth row: no, no, no.
To elaborate on the final point, note that there is a non-trivial linear combination
$$a(x-1) + b(x-1)^2 + c(x-1)(x+1) = 0,$$
where the zero on the right hand side mean "the zero polynomial," i.e. there are non-zero number a, b, c so that the function on the left hand side is zero *no matter what x is.* Indeed, take for example $a = 2, b = -1, c = 1$.
- This is an orthogonal matrix so the inverse is the transpose.

(3) This problem is similar to problem 2 in Spring 2014.

(4)
- We follow the Gram-Schmidt process. The vector $\vec{v}_1$ satisfies $\|\vec{v}_1\| = \sqrt{4} = 2$, so set
$$\vec{u}_1 = \begin{pmatrix} -1/2 \\ 1/2 \\ 1/2 \\ -1/2 \end{pmatrix}.$$
Then $\vec{v}_2^{\perp} = \vec{v}_2 - (\vec{v}_2 \cdot \vec{u}_1)\vec{u}_1 = \vec{v}_2 - 4\vec{u}_1$, so we get
$$\vec{v}_2^{\perp} = \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}.$$

The norm of $\vec{v}_2^\perp$ is 2, so setting

$$\vec{u}_2 = \frac{1}{2}\vec{v}_2^\perp = \begin{pmatrix} 1/2 \\ 1/2 \\ -1/2 \\ -1/2 \end{pmatrix},$$

we have that $\vec{u}_1$ and $\vec{u}_2$ orthonormalize $\vec{v}_1$ and $\vec{v}_2$.

- This can be done in several ways, but the quickest is to let

$$A = \begin{pmatrix} | & | \\ \vec{u}_1 & \vec{u}_2 \\ | & | \end{pmatrix}.$$

Then the matrix for the orthogonal projection is AA^T, which explicitly is

$$AA^T = \begin{pmatrix} -1/2 & 1/2 \\ 1/2 & 1/2 \\ 1/2 & -1/2 \\ -1/2 & -1/2 \end{pmatrix} \begin{pmatrix} -1/2 & 1/2 & 1/2 & -1/2 \\ 1/2 & 1/2 & -1/2 & -1/2 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}.$$

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