201 Linear Algebra, Practice Midterm Solutions

1. Row reduce the augmented matrix $\begin{pmatrix} 1 & 2 & 3 & 1 \\ 3 & 4 & 7 & 1 \\ 5 & 6 & 11 & 1 \end{pmatrix}$ to $\begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. Therefore the solution set can be described as vectors in $\mathbb{R}^3$ of the form $\begin{pmatrix} -1-t \\ 1-t \\ t \end{pmatrix}$ where $t \in \mathbb{R}$.

   The rank of the coefficient matrix = the number of leading 1's in the row-reduced echelon form = 2.

2. The coefficient matrix A in question 1. has Ker = span$\{\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}\}$, since rref$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$.

   This means that the columns vectors of A are not linearly independent. The elements of Ker$A$ correspond to linear dependency relations among the column vectors. Therefore $t\begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix} + t\begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix} - t\begin{pmatrix} 3 \\ 7 \\ 11 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ $(t \in \mathbb{R})$ are all the ways in which the column vectors of A are linearly dependent.

3. $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$. The matrix row reduces to I_3, therefore is invertible. Thorefore the linear transformation is invertible.

   Row reduce the augmnented matrix $\begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$ to $\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{pmatrix}$; just swap rows 1 and 3. Therefore the matrix for T^{-1}, which is the inverse of $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$, is $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$.

   Notice that $T^{-1} = T$.

4.

$$\text{proj}_L\vec{e}_1 = \frac{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1/3 \\ 1/3 \\ 1/3 \end{pmatrix}$$

$$\text{proj}_L\vec{e}_2 = \frac{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1/3 \\ 1/3 \\ 1/3 \end{pmatrix}$$

$$\text{proj}_L \vec{e}_3 = \frac{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1/3 \\ 1/3 \\ 1/3 \end{pmatrix}$$

Therefore the matrix of the this transformation is $\begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{pmatrix}$.

The image of this transformation is the span of the column vectors of this matrix, which is the line L.

The matrix row reduces to $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Therefore the kernel consist of vectors in $\mathbb{R}^3$ which are of the form $\{\begin{pmatrix} -s-t \\ s \\ t \end{pmatrix} | s, t \in \mathbb{R}\}$. This is the plane inside $\mathbb{R}^3$ consisting of all vectors perpendicular the line L.

5. (a) False. If a 4×4 matrix has rank 3, then it's row reduced echelon form looks like $\begin{pmatrix} 1 & 0 & 0 & k \\ 0 & 1 & 0 & l \\ 0 & 0 & 1 & m \\ 0 & 0 & 0 & 0 \end{pmatrix}$, for some scalars k, l and m. This matrix clearly has a nonzero kernel.

   (b) False. The first 3 equations (in 3 variables) can be consistent. The fourth equation can a linear combinatin of the first three equations. Think of an example.

   (c) False. $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

   (d) False. $T(\vec{e}_1) = 5\vec{e}_2$ and T is linear $\Rightarrow T(5\vec{e}_1) = 5T(\vec{e}_1) = 5.5.\vec{e}_2 \neq \vec{e}_2$.