SYSTEMS OF INHOMOGENOUS LINEAR ODES

We are interested in studying the following inhomogenous linear system

$$\mathbf{Y}' = A \cdot \mathbf{Y} + \mathbf{G}$$

where A is an $n \times n$ matrix and $\mathbf{G} = \mathbf{G}(t)$ is a $\mathbb{R}^n$ valued function (for now assume it is continuous – it will be important to consider more general situations, but we will return to this latter). Physically, this models how the *closed* system

$$\mathbf{Y}' = A \cdot \mathbf{Y}$$

reacts to some external "force" $\mathbf{G}$. This is clearest (and this will be our main model) in the *forced harmonic oscillator* describing the motion of a unit mass on a spring:

$$x'' + bx' + kx = f(t).$$

By Newton's law, f genuinely corresponds to an external force acting on the mass (e.g. the wind blowing on it or it being hit by a hammer). This of course is the same as solving

$$\mathbf{Y}' = \begin{pmatrix} 0 & 1 \\ -k & -b \end{pmatrix} + \begin{pmatrix} 0 \\ f(t) \end{pmatrix}$$

by the identification

$$\mathbf{Y} = \begin{pmatrix} x \\ x' \end{pmatrix}.$$

The main things we are going to attempt to understand is

(1) How to find explicit solutions?
(2) Can we understand how the system reacts to the external force? For instance, if we want the system to behave in a given way what is it possible to find an external force that accomplishes this? How does small changes in the external force change the behavior of the system?
(3) A particularly, important example of the latter is concept of *resonance*. Resonance occurs when the forcing is at the same frequency as the natural one of the system and leads to qualitatively (and sometimes disastrously so) different behavior.

1. General Concepts

Notice that if $\mathbf{Y}_i(t)$ $(i = 1, 2)$ satisfies

$$\mathbf{Y}_i' = A \cdot \mathbf{Y}_i + \mathbf{G}_i$$

then

$$\mathbf{Y}_3 = \alpha \mathbf{Y}_1 + \beta \mathbf{Y}_2$$

satisfies (you should check this!)

$$\mathbf{Y}_3' = A \cdot \mathbf{Y}_3 + \alpha \mathbf{G}_1 + \beta \mathbf{G}_2.$$

This means that in general the space of solutions of an inhomogenous linear system form an *affine space* rather than a *linear space*, that is, one *cannot* add two solutions to the problem together and get a new solution. However, one *can* add a solution to the inhomogenous problem to any solution to the corresponding *homogenous* problem

$$\mathbf{Y}' = A \cdot \mathbf{Y}$$

to obtain a new solution of the inhomogenous problem. In other words, to find a general solution to

$$\mathbf{Y}' = A \cdot \mathbf{Y} + \mathbf{G}$$

one finds *one* solution, called a *particular solution*, $\mathbf{Y}_P$ to the system and the general solution, $\mathbf{Y}_H$ to the the homogenous system, which we know how to find (and in pricinciple can actually compute out). That is, the general solution to the inhomogenous system is

$$\mathbf{Y}(t) = \mathbf{Y}_N + \mathbf{Y}_P = e^{tA} \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} + \mathbf{Y}_P.$$

Note that the particular solutions is not unique (since one can add any solution to homogenous problem to it). Often one takes it to be the unique solution to the IVP

$$\begin{cases} \mathbf{Y}' = A \cdot \mathbf{Y} + \mathbf{G} \\ \mathbf{Y}(0) = 0 \end{cases}$$

as then one can solve the IVP

$$\begin{cases} \mathbf{Y}' = A \cdot \mathbf{Y} + \mathbf{G} \\ \mathbf{Y}(0) = \mathbf{Y}_0 \end{cases}$$

as

$$\mathbf{Y}(t) = e^{tA}\mathbf{Y}_0 + \mathbf{Y}_P.$$

For the forced harmonic oscillator, it is enough to find a particular solution x_P satisfying

$$x_P'' + bx_P' + kx_P = f(t)$$

as then the particular solution is

$$\mathbf{Y}_P = \begin{pmatrix} x_P \\ x_P' \end{pmatrix}$$

and it is often simpler to find x_P then the corresponding $\mathbf{Y}_P$.

2. Undetermined coefficients

A tried and true method of finding solutions to (any) equation is to make a guess at the form of the solution (called an *ansatz*) together with some parameters and then plug it into the equation to see if you can find a solution by changing the parameters.

In the context of the inhomogenous linear systems the things to keep in mind is that:

(1) If each entry of $\mathbf{Y}$ is a polynomial of degree n, then each entry of $\mathbf{Y}' - A \cdot \mathbf{Y}$ is a polynomial of degree at most n;
(2) If each entry of $\mathbf{Y}$ is a linear comination of $e^{\lambda_i t}$, then each entry of $\mathbf{Y}' - A \cdot \mathbf{Y}$ is a linear combination of $e^{\lambda_i t}$ (some entries may be zero!);

(3) If each entry of $\mathbf{Y}$ is a linear comination of $e^{a_i t}\cos\omega_i t$ and $e^{a_i t}\sin\omega_i t$, then each entry of $\mathbf{Y}' - A\cdot\mathbf{Y}$ is a linear combination of the same form (again some entries may be zero!) (this is just a case of the former by using Euler's formula and complex λ_i)
(4) Multiplying, examples from 2) or 3) by a polynomial is also well behaved.

This gives a method to try and solve inhomogenous problems when $\mathbf{G}(t)$ is a sum of vectors of one of the preceding forms one just guesses $\mathbf{Y}(t)$ appropriately.

Example 2.1. Find a particular solution to

$$y' - y = t\cos t.$$

To do this we make the ansatz

$$y_p = at\cos t + b\cos t + ct\sin t + d\sin t$$

plugging this in we obtain (after some work)

$$-(a+c)t\sin t + (a+d-b)\cos t + (c-b-d)\sin t + (c-a)t\cos t = t\cos t$$

which implies $a = -1/2, c = 1/2, d = 1/2, b = 0$.

Hence,

$$y_p = -\frac{1}{2}t\cos t + \frac{1}{2}t\sin t + \frac{1}{2}\sin t$$

is a particular solution.

Some care has to be taken when $\mathbf{G}(t)$ itself solves the homogenous equation. To see this consider

Example 2.2. Find a particular solution to for $a \neq -1$.

$$y' + ay = e^t$$

. We make the ansatz $y_p = be^t$ which pluggin in gives

$$be^t + abe^t = e^t$$

that is, $b = \frac{1}{1+a}$ which is fine except when $a = -1$. That is,

$$y_p = \frac{1}{a+1}e^t$$

is a particular solution except when $a = -1$.

When $a = -1$ we are at a situation where the forcing is the same as the natural "frequency" of the system. To remedy this, we note that if

$$\mathbf{G}(t)$$

solves the homogenous problem, then one can (straightforwardly check) that the particular solution is given by

$$\mathbf{Y}_P = t\mathbf{G}(t)$$

For instance, one checks that

$$y_p = te^t$$

is a particular solution to

$$y' - y = e^t$$

from the previous problem.

This is a first example of resonance phenomena.

3. Variation of parameters

While for many problems the method of undetermined coefficients works (and is the most practical method to find a solution) it will be nice to have a more "systematic" approach.

Lets start with the one-dimensional setting

$$y' = ay + g(t)$$

multiplying both sides by e^{-at} we see that that this equaiton is the same as

$$\left(e^{-at}y\right)' = e^{-at}g(t)$$

Hence, integrating (and using the Fundamental theorem of calculus – need g to be continuous to justify this step) gives the particular solution

$$y_P(t) = e^{at}\int_0^t e^{-as}g(s)ds = \int_0^t e^{(t-s)a}g(s)ds$$

which has initial condition $y_P(0)$ and so solves the IVP

$$\begin{cases} y' = ay + g \\ y(0) = 0 \end{cases}$$

Hence, the general solution is given by

$$y(t) = Ca^{at} + \int_0^t e^{(t-s)a}g(s)dsf$$

More generally, we have the following

Theorem 3.1. *For continuous* $\mathbf{G}$*, the solution to the IVP*

$$\begin{cases} \mathbf{Y}' = A \cdot \mathbf{Y} + \mathbf{G} \\ \mathbf{Y}(0) = \mathbf{Y}_0 \end{cases}$$

is given by

$$\mathbf{Y}(t) = e^{tA}\mathbf{Y}_0 + \int_0^t e^{(t-s)A}\mathbf{G}(s)ds$$

The advantage of this approach is that it reduces the problem to an integral. The disadvantage is that it can be cumbersome to compute the integrand (since we have to solve the homogenous ODE for a lot of different initial conditions). One is also not necessarily able to integrate the answer into a closed form (though this is not so much of a drawback in practice).

Example 3.2. Find a particular solution to

$$x'' - x = t^2$$

using variation of parameters.

First we convert to the equivalent first order system

$$\mathbf{Y}' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 \\ t^2 \end{pmatrix} = A\mathbf{Y} + \mathbf{G}$$

We recognize that A has eigenvalues ± 1, and work out that it is diagonalized by

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = T^{-1}AT = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

Hence,

$$e^{tA} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}$$

Hence, a particular solution to the first order equation is given by

$$\begin{aligned} \mathbf{Y}_P(t) &= e^{tA} \int_0^t e^{-sA} \mathbf{G}(s) \\ &= \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix} \int_0^t \begin{pmatrix} \cosh(-s) & \sinh(-s) \\ \sinh(-s) & \cosh(-s) \end{pmatrix} \begin{pmatrix} 0 \\ s^2 \end{pmatrix} ds \\ &= \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix} \int_0^t \begin{pmatrix} -s^2 \sinh s \\ s^2 \cosh s \end{pmatrix} ds \\ &= \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix} \begin{pmatrix} -(t^2+2)\cosh t + 2t\sinh(t) \\ (t^2+2)\sinh(t) - 2t\cosh(t) \end{pmatrix} \\ &= \begin{pmatrix} t^2+2 \\ 2t \end{pmatrix} \end{aligned}$$

Hence a particular solution to the second order equation is

$$x_P(t) = t^2 + 2$$

(Now try finding this by undetermined coefficients!)

4. Discontinuous Forcing

One of the nice things about variation of parameters is that we can use it to *define* solutions to inhomogenous ODEs with forcings $\mathbf{G}(t)$ which are not continuous (or even functions). Rather than make this completely precise and rigorous and as general as possible (which is beyond the scope of the course in any case) we will restrict attention to two cases:

(1) The function is *piecewise continuous*. That is, there exists a $T > 0$, so that for each $t \in \mathbb{R}$, $\mathbf{G}(t)$ has a only a finite number of jump discontinuities on $[t, t+T]$ and no other discontinuities;
(2) $\mathbf{G}(t) = \mathbf{V}\delta_{t_0}$ for a fixed vector $\mathbf{V}$ and δ_{t_0} is the *Dirac delta* with unit mass at $t = t_0$ (we will define this shortly).

Obviously, if $\mathbf{G}$ has finitely many jump discontinuities then it is piecewise continuous. Likewise, if $\mathbf{G}(t)$ is periodic with period T (i.e. $\mathbf{G}(t+T) = \mathbf{G}(t)$ for all $t \in [0, T)$) and has a finite number of jump discontinuities on $[-T, T]$ and is otherwise continuous, then it is piecewise continuous.

4.1. Jump discontinuities.

Recall, that a function $\mathbf{G}$ has a jump discontinuity at $t = t_0$, if the left and right limits exist (but don't agree) and are finite. That is,

$$\lim_{t \to t_0^+} \mathbf{G}(t) = \mathbf{V}^+ \in \mathbb{R}^n$$

and

$$\lim_{t \to t_0^-} \mathbf{G}(t) = \mathbf{V}^- \in \mathbb{R}^n$$

we will sometimes write $\mathbf{G}(t_0^{\pm})$ for $\mathbf{V}^{\pm}$.

Let us consider a real-valued function $\mathbf{G}(t)$ which is continuous except at $t = 1$ where the solution has a jump discontinuity. What is a solution to

$$\mathbf{Y}' = A \cdot \mathbf{Y} + \mathbf{G}(t),$$

supposed to mean? The natural way to make sense of this (which you implicitly were asked to do in homework early in the course) is to solve the well-posed problem on each side of $t = 0$ separately and then match solutions at $t = 0$ so that the the resulting function is actually continuous. Call this the *matching* solution.

Specifically, suppose that

$$\mathbf{G} = \begin{cases} \mathbf{G}_+(t) & t > 1 \\ \mathbf{G}_-(t) & t < 1 \end{cases}$$

has a jump discontinuity at $t = t_0$. We can always assume that $\mathbf{G}_{\pm}$ are actually continuous functions for all $t \in \mathbb{R}$ (though we only care about their values on the restricted sense). Its not hard to check this if you are interested at least for real valued $G_{\pm}$.

Solving the IVP for all $t \in \mathbb{R}$

$$\begin{cases} \mathbf{Y}' = A \cdot \mathbf{Y} + \mathbf{G}(t) \\ \mathbf{Y}(0) = \mathbf{Y}_0 \end{cases}$$

consists of solving on $(-\infty, 1)$

$$\begin{cases} \mathbf{Y}' = A \cdot \mathbf{Y} + \mathbf{G}_-(t) \\ \mathbf{Y}(0) = \mathbf{Y}_0 \end{cases}$$

Denote, the solution by $\mathbf{Y}^-$. Letting

$$\mathbf{Y}_1 := \lim_{t \to 1^-} \mathbf{Y}^-(t)$$

which exists due to the nature of the ODE. As we assume $\mathbf{G}^-$ is continuous on all of $\mathbb{R}$, so is $\mathbf{Y}^-$ and so we have the simplification that

$$\mathbf{Y}_1 = \lim_{t \to 1^-} \mathbf{Y}^-(t)\mathbf{Y}^-(1)$$

Continue by solving on $(1, \infty)$ the "IVP"

$$\begin{cases} \mathbf{Y}' = A \cdot \mathbf{Y} + \mathbf{G}_+(t) \\ \lim_{s \to 1^+} \mathbf{Y}(s) = \mathbf{Y}_1 \end{cases}$$

Again, because solutions will be continuous (again because we assume $\mathbf{G}^+$ is) one can simplify this and actually solve

$$\begin{cases} \mathbf{Y}' = A \cdot \mathbf{Y} + \mathbf{G}_+(t) \\ \mathbf{Y}(1) = \mathbf{Y}_1 \end{cases}$$

In either case, let $\mathbf{Y}^+$ be the solution. Finally, one defines the value of the matching solution *at* $t = 1$ to be $\mathbf{Y}_1$. Hence, the solution is

$$\mathbf{Y}(t) = \begin{cases} \mathbf{Y}^-(t) & t < 1 \\ \mathbf{Y}_1 & t = 1 \\ \mathbf{Y}^+ & t > 1 \end{cases}$$

You should convince yourself that this is exactly the same as the solution given by variation of parameters, i.e., the solution above can be represented as

$$\mathbf{Y}(t) = e^{tA}\mathbf{Y}_0 + \int_0^t e^{(t-s)A}\mathbf{G}(s)ds.$$

Once, we've checked this for this for a single jump discontinuity as in the above example it is not hard to see that it also holds for finitely many of them (or indeed infinitely many as long as there are only finitely many on each interval of a fixed size – i.e., for piecewise continuous functions).

To summarize

Theorem 4.1. *If there is a* $T > 0$*, so that* $\mathbf{G}(t)$ *is piecewise continuous, then the solution to the IVP*

$$\begin{cases} \mathbf{Y}' = A \cdot \mathbf{Y} + \mathbf{G}(t) \\ \mathbf{Y}(t_0) = \mathbf{Y}_0 \end{cases}$$

given by the variation of parameters formula agrees with the matching solution – i.e., the solution given by solving on each interval where $\mathbf{G}$ *is continuous and matching values at the endpoints of the interval. Furthermore, this solution is continuous.*

Example 4.2. Consider the *Heaviside* function

$$H(t) = \begin{cases} 0 & t \leq 0 \\ 1 & t > 0 \end{cases}$$

and let $H_a(t) = H(t-a)$ be the Heaviside function that "turns on" at $t = a$. Find a particular solution to the equation

$$y' = 2y + H_a(t).$$

Using the variation of parameters formula, we have

$$y_P(t) = e^{2t} \int_a^t e^{-2s} H(s)ds = \begin{cases} 0 & t \leq a \\ -\frac{1}{2} + \frac{1}{2}e^{2(t-a)} & t > a. \end{cases}$$

(Note that it doens't matter where we start the integral from – it just corresponds to adding a different solution to the homogenous equation.) Observe this is continuous at $t = 0$. We may rewrite y_P as

$$y_P(t) = -\frac{1}{2}(1 - e^{2(t-a)})H_a(t).$$

To find a particular solution by matching, first note that a valid solution on $(-\infty, a)$ is given by solving

$$y' = 2y$$

and such a solution is $y_P(t) = 0$ for $t < a$. The limit $t \to a^-$ of this solution is obviously 0. Hence, to match in (a, ∞) we must solve the IVP

$$\begin{cases} y' = 2y + 1 \\ y(a) = 0 \end{cases}$$

whcih is easily seen (by variation of paramters) to be given by $\frac{1}{2}\left(e^{2(t-a)} - 1\right)$ and so $y_P(t) = \frac{1}{2}\left(e^{2(t-a)} - 1\right)$ – agreeing with the previous computation.

Notice that the formula of variation of parameters gives us the following useful fact which sometimes simplifies computations.

Theorem 4.3. *Suppose there is a $T > 0$, so that $\mathbf{G}(t), \mathbf{G}_1(t)$ and $\mathbf{G}_2(t)$ are piecewise continuous. If $\mathbf{G}(t) = \mathbf{G}_1(t) + \mathbf{G}_2(t)$ and $\mathbf{Y}_P^1$ and $\mathbf{Y}_P^2$ are particular solutions to the respective systems*

$$\mathbf{Y}' = A \cdot \mathbf{Y} + \mathbf{G}_1(t) \text{ and } \mathbf{Y}' = A \cdot \mathbf{Y} + \mathbf{G}_2(t)$$

then $\mathbf{Y}_P = \mathbf{Y}_P^1 + \mathbf{Y}_P^2$ is a solution to the system

$$\mathbf{Y}' = A \cdot \mathbf{Y} + \mathbf{G}(t).$$

Example 4.4. Let

$$I(t) = H_0(t) - H_1(t)$$

be a unit impulse. Define the square wave of period 2 by

$$S(t) = \sum_{n=-\infty^\infty} I(t + 2n).$$

Find a particular solution to

$$y' = 2y + S(t).$$

We start by finding a particular solution to

$$y' = 2y + I(t)$$

Using the above theorem and the previous example we obtain

$$\begin{aligned} y_P^0(t) &= -\frac{1}{2}(1 - e^{2t})H_0(t) + \frac{1}{2}(1 - e^{2(t-1)})H_1(t) \\ &= -\frac{1}{2}I(t) + \frac{1}{2}e^{2t}\left(H_0(t) - e^{-2}H_1(t)\right). \end{aligned}$$

You should convince yourself, that a particular solution to

$$y' = 2y + I(t + 2n).$$

is

$$\begin{aligned} y_P^n(t) &= -\frac{1}{2}I(t + 2n) + \frac{1}{2}e^{2(t+2n)}\left(H_0(t + 2n) - e^{-2}H_1(t + 2n)\right) \\ &= -\frac{1}{2}I(t + 2n) + \frac{1}{2}e^{2(t+2n)}\left(H_{-2n}(t) - e^{-2}H_{1-2n}(t)\right). \end{aligned}$$

Hence, a particular solution of the original ODE is

$$y_P(t) = \sum_{n=-\infty}^{\infty} y_P^n(t)$$

I'll leave it to you to sum this series and obtain a closed form for the solution. Note that, technically, we have to justify taking an infinite sum, but observe that for each t at most one summand is non-zero so this is not an actual issue.

4.2. **Dirac delta.** The Dirac delta, δ_{t_0}, is a bit harder to understand because it is not really a function. The motivation for the Dirac delta, is the consideration of the following functions which will "model" transmission of a unit force over an interval of time 2Δ which is very short. Mathematically, this consists of a choice of functions $D_\Delta(t)$ with $\Delta > 0$ which are

(1) Non-negative (i.e., push only in one direction);

(2) Zero outside of $[-\Delta, \Delta]$ (i.e., push only for a short amount of time)
(3) Have integral 1 (i.e., transmit only a unit amount of force)
(4) Are "reasonable," (e.g., have a at most two jump discontinuities and no other discontinuities).

In general, in the limit as $\Delta \to 0$ the particular shapes of D_Δ we choose (i.e., the precise shape of the impulse) won't matter, though it is beyond the scope of the course to show this.

For simplicity, we consider the following very simple possible choice

$$D_\Delta(t) = \begin{cases} \frac{1}{2}\Delta^{-1} & t \in [-\Delta, \Delta] \\ 0 & \text{otherwise.} \end{cases} = \frac{1}{2}\Delta^{-1}\left(H_{-\Delta}(t) - H_\Delta(t)\right)$$

Clearly,

$$\int_{-\infty}^{\infty} D_\Delta(t)dt = \frac{1}{2}\Delta^{-1}(\Delta - (-\Delta)) = 1$$

and in fact, for any continuous function f,

$$\int_{-\infty}^{\infty} f(t)D_\Delta(t)dt = f(0) + o(\Delta) \tag{4.1}$$

We now "define" the Dirac delta at $t = 0$, to be the "limit"

$$\delta_0 = \lim_{\Delta \to 0} D_\Delta(t).$$

While, this *isn't* a function, we will be able to handle it (in many ways) as if it were. The key fact is that based on the observation (4.1), we "morally" have that for any continuous function

$$\int_{-\infty}^{\infty} f(t)\delta_0(t)dt = f(0).$$

This is the *only* way you should think of the δ_0. With this in mind, we are able to generalize this to the symbol $\delta_{t_0}(t)$ for $t_0 \in \mathbb{R}$ which satisfying (for continuous f)

$$\int_{-\infty}^{\infty} f(t)\delta_{t_0}(t)dt = f(t_0).$$

Notice, that we can add Dirac delta functions together and multiple them by real number a in a consistent manner. So, for instance, (for continuous f)

$$\int_{-\infty}^{\infty} f(t)(a\delta_{t_0}(t)+b\delta_{t_1}(t))dt = af(t_0)+bf(t_1) = a\int_{-\infty}^{\infty} f(t)\delta_{t_0}(t)dt+b\int_{-\infty}^{\infty} f(t)\delta_{t_1}(t)dt.$$

Another important property

$$\delta_{t_0}(t) = \delta_0(t - t_0).$$

Now the key fact is that the above observations mean that we can use the variation of parameters formula to make sense of the solutions to the ODEs

$$\mathbf{Y}' = A \cdot \mathbf{Y} + \mathbf{V}\delta_{t_0}$$

away from $t = t_0$ (trying to make sense of things at $t = t_0$ leads to lots of ambiguity – and so must be treated with care).

Namely, one can define the particular solution which is zero at $t = t_*$ (when $t_* \neq t_0$) by

$$\mathbf{Y}_P = e^{tA}\int_{t_*}^{t} e^{-sA}\mathbf{V}\delta_{t_0}ds$$

Using the properties of the Dirac delta, we see that if $t_* < t_0$, then this can be simplified to

$$\mathbf{Y}_P = \begin{cases} 0 & t < t_0 \\ e^{(t-t_0)A}\mathbf{V} & t > t_0. \end{cases} = e^{(t-t_0)A}\mathbf{V}H_{t_0}(t)$$

and if $t_* > t_0$ this can be simplified to

$$\mathbf{Y}_P = \begin{cases} 0 & t > t_0 \\ -e^{(t-t_0)A}\mathbf{V} & t > t_0. \end{cases} = -e^{(t-t_0)A}\mathbf{V}(1 - H_{t_0}(t)).$$

It is important to not get confused by the meaning at $t = t_0$. Despite, the use of the Heaviside function (which is technically 0 at $t = t_0$) the solution is (in general) meaningless at $t = t_0$ and not 0.

In either case, the solution has a jump discontinuity at $t = t_0$ whenever $\mathbf{V} \neq 0$. And jumps from the zero solution to the homogenous ODE to the (non-zero) solution of the IVP

$$\begin{cases} \mathbf{Y}' = A \cdot \mathbf{Y} \\ \mathbf{Y}(t_0) = \mathbf{V} \end{cases}$$

We can also take this as a definition.

Example 4.5. Solve for $a \in \mathbb{R}$

$$\begin{cases} y' = a\delta_{t_0}(t) \\ y(t_0 - 1) = 0 \end{cases}$$

Its clear from the above discussion that $y(t) = aH_{t_0}(t)$ is the solution.

Notice, this allows us to interpret $a\delta_{t_0}$ as the derivative of $aH_{t_0}(t)$ (which is obviously not differentiable at 0 in any normal sense when $a \neq 0$). This is a powerful perspective, but care should be taken.

While this all may seem completely arbitrary it is in fact quite naturally justified.

Theorem 4.6. *For $t_* \neq t_0$, let $\mathbf{Y}_\Delta$ solve the IVP*

$$\begin{cases} \mathbf{Y}' = A \cdot \mathbf{Y} + \mathbf{V}D_\Delta(t - t_0) \\ \mathbf{Y}(t_*) = 0 \end{cases}$$

then, as $\Delta \to 0$, these solutions converge (away from t_0) to the solution to

$$\begin{cases} \mathbf{Y}' = A \cdot \mathbf{Y} + \mathbf{V}\delta_{t_0} \\ \mathbf{Y}(t_*) = 0 \end{cases}$$

Example 4.7. Consider the IVP

$$\begin{cases} y' = aD_\Delta(t - t_0) \\ \mathbf{Y}(t_0 - 1) = 0 \end{cases}$$

and show that as $\Delta \to 0$ the solutions y_Δ converge, away from $t = t_0$, to $y(t) = aH_{t_0}(t)$ the solution to

$$\begin{cases} y' = a\delta_{t_0}(t) \\ y(t_0 - 1) = 0 \end{cases}$$

Its not hard to determine that for $\Delta < 1$ that

$$y_\Delta(t) = \frac{a}{2\Delta}(t - (t_0 - \Delta))\left(H_{-\Delta+t_0} - H_{\Delta+t_0}\right) + aH_{\Delta+t_0}.$$

Indeed, on each interval $(-\infty, -\Delta + t_0)$, $(-\Delta + t_0, \Delta + t_0)$ and $(\Delta + t_0, \infty)$ this is the obvious solution and since the resulting y_Δ is continuous, it is the matching solution. Clearly, for $t \neq t_0$, this converges as desired.

While it this does converges at $t = t_0$ as well (to $\frac{1}{2}a$) this is misleading, since if one changed the shape of the impulse, one could get different values. Its beyond the scope of the course to discuss this in detail so it is best to leave it undefined.

One might wonder what happens if $t_* = t_0$? There is not a single consistent choice. Since we are usually interested in the evolution *forward* in time, we will consider the solution to the IVP

$$\begin{cases} \mathbf{Y}' = A \cdot \mathbf{Y} + \mathbf{V}\delta_{t_0} \\ \mathbf{Y}(t_0) = 0 \end{cases}$$

as

$$\mathbf{Y}_P(t) = \lim_{s \to 0^-} \mathbf{Y}_s(t)$$

where $\mathbf{Y}_s(t)$ solves

$$\begin{cases} \mathbf{Y}' = A \cdot \mathbf{Y} + \mathbf{V}\delta_{t_0} \\ \mathbf{Y}(t_0 + s) = 0 \end{cases}$$

You should convince yourself that this is the same as

$$\mathbf{Y}_P = \begin{cases} 0 & t < t_0 \\ e^{(t-t_0)A}\mathbf{V} & t > t_0. \end{cases} = H_{t_0}(t)e^{(t-t_0)A}\mathbf{V}.$$

That is, the solution which is initiated by an instantaneous pulse. Where again the value at $t = t_0$ is should be thought of as not well defined. If we want to distinguish this IVP from the "backwards" one we will denote it by

$$\begin{cases} \mathbf{Y}' = A \cdot \mathbf{Y} + \mathbf{V}\delta_{t_0} \\ \mathbf{Y}(t_0^-) = 0 \end{cases}$$

Example 4.8. Solve

$$\begin{cases} y'' + y = a\delta_0(t) \\ y(-1) = 0, y'(-1) = 0 \end{cases}$$

And call this solution y_0. Physically, this solution describes the motion of a spring after being hit by an "ideal" hammer blow.

Justify this by solving (for $\Delta < 1$)

$$\begin{cases} y'' + y = aD_\Delta(t) \\ y(-1) = 0, y'(-1) = 0 \end{cases}$$

and showing that as $\Delta \to 0$, these solutions, y_Δ converge (away from 0) to y_0.

First of all, we convert the first equation to a first order system (so we can use variation of parameters). We are then considering

$$\begin{cases} \mathbf{Y}' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + \begin{pmatrix} 0 \\ a \end{pmatrix} \delta_0(t) \\ \mathbf{Y}(-1) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{cases}$$

As

$$e^{tA} = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$$

we obtain that the solution to this is

$$\begin{aligned} \mathbf{Y}_0 &= e^{tA} \int_{-1}^{t} e^{-sA} \begin{pmatrix} 0 \\ a \end{pmatrix} \delta_0(s) ds \\ &= e^{tA} \int_{-1}^{t} \begin{pmatrix} -a \sin s \\ a \cos s \end{pmatrix} \delta_0(s) ds \\ &= e^{tA} H_0(t) \begin{pmatrix} -a \sin 0 \\ a \cos 0 \end{pmatrix} \\ &= e^{tA} H_0(t) \begin{pmatrix} 0 \\ a \end{pmatrix} \\ &= H_0(t) \begin{pmatrix} a \sin t \\ a \cos t \end{pmatrix} \end{aligned}$$

Hence,

$$y_0(t) = a H_0(t) \sin t = \begin{cases} 0 & t < 0 \\ a \sin t & t > 0 \end{cases}$$

Notice, that $\mathbf{Y}_0$ is discontinuous at $t = 0$, but y_0 is continuous at $t = 0$.

Converting, now the second equation to a first order system we obtain solutions

$$\begin{aligned} \mathbf{Y}_\Delta(t) &= e^{tA} \int_{-1}^{t} e^{-sA} \frac{1}{2} \Delta^{-1} \left(H_{-\Delta}(s) - H_\Delta(s) \right) \begin{pmatrix} 0 \\ a \end{pmatrix} ds \\ &= \frac{1}{2} \Delta^{-1} e^{tA} \int_{\min\{t,-\Delta\}}^{\min\{t,\Delta\}} \begin{pmatrix} -a \sin s \\ a \cos s \end{pmatrix} ds \\ &= \frac{a}{2} \Delta^{-1} e^{tA} \begin{cases} \begin{pmatrix} 0 \\ 0 \end{pmatrix} & t < -\Delta \\ \begin{pmatrix} \cos t - \cos \Delta \\ \sin t + \sin \Delta \end{pmatrix} & -\Delta < t < \Delta \\ \begin{pmatrix} 0 \\ 2 \sin \Delta \end{pmatrix} & t > \Delta \end{cases} \\ &= \frac{a}{2} \Delta^{-1} \begin{cases} \begin{pmatrix} 0 \\ 0 \end{pmatrix} & t < -\Delta \\ \begin{pmatrix} 1 - \cos \Delta \cos t + \sin t \sin \Delta \\ \cos \Delta \sin t + \sin \Delta \cos t \end{pmatrix} & -\Delta < t < \Delta \\ \begin{pmatrix} 2 \sin \Delta \sin t \\ 2 \sin \Delta \cos t \end{pmatrix} & t > \Delta \end{cases} \end{aligned}$$

Hence,

$$y_\Delta(t) = \frac{a}{2} \Delta^{-1} \begin{cases} 0 & t < -\Delta \\ 1 - \cos \Delta \cos t + \sin t \sin \Delta & -\Delta < t < \Delta \\ 2 \sin \Delta \sin t & t > \Delta \end{cases}$$

Letting, $\Delta \to 0$, we see tha we have

$$\lim_{\Delta\to 0} y_\Delta(t) = \begin{cases} 0 & t > 0 \\ a\sin t & t > 0 \end{cases} = aH_0(t)\sin t$$

where we used that $\lim_{\Delta\to 0} \frac{\sin\Delta}{\Delta} = 1$. This verifies the claim.

Finally, we note that the following simplification for second order ODEs

Theorem 4.9. *For $t_* < t_0$, a solution to the IVP*

$$\begin{cases} y'' + by' + ky = a\delta_{t_0}(t) \\ y(t_*) = y'(t_*) = 0 \end{cases}$$

is given by

$$y(t) = \hat{y}(t)H_{t_0}(t)$$

where $\hat{y}$ solves the IVP

$$\begin{cases} y'' + by' + ky = 0 \\ y(t_0) = 0 \\ y'(t_0) = a. \end{cases}$$

Notice that y is continuous, but its derivative has a jump discontinuity at $t = t_0$. Hence, in this case, we can extend the solution to $t = t_0$ in an unambiguous way (i.e., by 0).

5. Laplace Transform

We now turn to the *Laplace transform.* Roughly speaking, the Laplace transform is an operation that converts ODE problems into algebraic problems. In many cases, this allows one to more easily see what is going on in the problem. With this somewhat vague motivation in mind, lets start by defining what we mean and checking some simple properties.

5.1. **Definition.**

We begin by defining the transform and computing it for some simple functions.

Definition 5.1. For a piecewise continuous function $f(t)$ we define the Laplace transform $\mathcal{L}\{f(t)\}$ of f to be the function of a real variable s given by the improper integral

$$\mathcal{L}\{f(t)\}(s) := \int_0^\infty e^{-st} f(t)dt = \lim_{k\to\infty} \int_0^k e^{-st} f(t)dt.$$

We restrict the domain of the function $\mathcal{L}\{f(t)\}(s)$ to be those values s so that the improper integral converges – this is considered part of the information of the Laplace transform. We will often write $F(s)$ for $\mathcal{L}\{f(t)\}(s)$.

The domain of f (i.e., the variable t) is called the *time domain* or *t-domain*; while the domain of F (i.e., the variable s) is called *frequency domain* or *s-domain.* While we will only very briefly touch on this, the above definition is (by Euler's formula) perfectly well defined for *complex* values of s. The function F is then a complex valued function of a complex variable s. Complex analyis techniques, then allows one to interpret the value of the Laplace transform at almost every value of s (even where the integral diverges). The values of s where this generalized Laplace transform doesn't have a well defined value carry important information about the function – we will see this in practice, but won't be able to develop the theory.

Example 5.2. Let $f(t) = e^{at}$ and denote its Laplace transform by $F(s)$. We compute

$$\begin{aligned} F(s) &= \int_0^\infty e^{-st} e^{at} dt \\ &= \lim_{k\to\infty} \frac{e^{(a-s)k} - e^0}{a-s} \\ &= \begin{cases} \frac{1}{s-a} & a < s \\ \infty & a \geq s \end{cases} \end{aligned}$$

That is, the Laplace transform of f is $F(s) = \frac{1}{s-a}$ with domain (a, ∞).

A special case of this (with $a = 0$) is

$$\mathcal{L}\{1\}(s) = \frac{1}{s}$$

for $s \in (0, \infty)$. This formula also holds for complex values of a, though now we need to restrict s so that $s \geq \operatorname{Re} a$.

Notice also, that as a function of s, such formula make sense (even though the integral does not) for all values of $a \neq s$. Furthermore, the real part of a carries the exponential growth (or decay) of the original function y while the imaginary part carries the "periodicity" of y. This turns out to be a very general phenomena.

5.2. **Properties of the Laplace transform.** Generally speaking, one does not compute the Laplace transform by direct integration. Instead one uses various properties (which are usally consequences of properties of integrals) along with a handy table. One thing we point out is that for must functions of interest, the domain of the Laplace transform is of the form (a, ∞).

An easy consequence of the linearity of integration is the following fact:

Proposition 5.3. *Let $f(t)$ and $g(t)$ be piecewise continuous functions with Laplace transforms $F(s)$ (with domain $s > a$) and $G(s)$ (with domian $s > b$), then*

$$\mathcal{L}\{af(t) + bg(t)\} = aF(s) + bG(s)$$

holds for $s \geq \max\{a, b\}$.

Example 5.4. Let $f(t) = \sin \omega t$. The Laplace transform, $F(s)$, of f can be computed by using that $\sin t = \frac{1}{2i}\left(e^{i\omega t} - e^{-i\omega t}\right)$ and our earlier computation. That is,

$$F(s) = \frac{1}{2i}\left(\mathcal{L}\{e^{i\omega t}\} - \mathcal{L}\{e^{-i\omega t}\}\right) = \frac{1}{2i}\left(\frac{1}{s-i\omega} - \frac{1}{s+i\omega}\right) = \frac{\omega}{s^2+\omega^2}.$$

Which holds, for $\omega > 0$.

Another elementary fact

Proposition 5.5. *If $f(t)$ is piecewise linear and has Laplace transform $F(s)$ with domain (a, ∞), then*

$$\mathcal{L}\{e^{bt} f(t)\} = F(s-b)$$

with domain $(a+b, \infty)$

Proof.

$$\begin{aligned}\mathcal{L}\{e^{bt}f(t)\} &= \int_0^\infty e^{-st}e^{bt}f(t)dt \\ &= \int_0^\infty e^{-(s-b)t}f(t)dt \\ &= F(s-b).\end{aligned}$$

□

A consequence of the change of variables formula is

Proposition 5.6. *If $f(t)$ is piecewise linear and has Laplace transform $F(s)$ with domain (a,∞), then for $\omega > 0$*

$$\mathcal{L}\{f(\omega t)\} = \omega^{-1}F(\omega^{-1}s)$$

with domain (a,∞)

Proof.

$$\begin{aligned}\mathcal{L}\{f(\omega t)\} &= \lim_{k\to\infty}\int_0^k e^{-st}f(\omega t)dt \\ &= \lim_{k\to\infty}\int_0^{\omega k} e^{-s\omega^{-1}u}f(u)\omega^{-1}du \\ &= \omega^{-1}F(\omega^{-1}s)\end{aligned}$$

□

Similarly, the fundamental theorem of calculus gives that

Proposition 5.7. *If $f(t)$ is differentiable with Laplace transform $F(s)$ and $|f(t)| \leq Ce^{at}$ for some constant C, then for $s > a$*

$$\mathcal{L}\{f'(t)\}(s) = sF(s) - f(0)$$

Proof. One computes that

$$\begin{aligned}\mathcal{L}\{f'(t)\}(s) &= \lim_{k\to\infty}\int_0^k e^{-st}f'(t)dt \\ &= \lim_{k\to\infty}\int_0^k \frac{d}{dt}\left(e^{-st}f(t)\right) + se^{-st}f(t)dt \\ &= \lim_{k\to\infty}\left(e^{-sk}f(k) - f(0) + s\int_0^k e^{-st}f(t)dt\right)\end{aligned}$$

Hence, taking the limit as $k \to \infty$ and using that $|e^{-sk}f(k)| \leq Ce^{(a-s)k}$ goes to zero as $k \to \infty$. Thus,

$$\mathcal{L}\{f'(t)\}(s) = sF(s) - f(0).$$

□

Example 5.8. We compute

$$\begin{aligned}\mathcal{L}\{\cos\omega t\} &= \mathcal{L}\{\omega^{-1}\frac{d}{dt}\sin\omega t\}\\ &= s\omega^{-1}\mathcal{L}\{\sin\omega t\} + \omega^{-1}\sin 0\\ &= \frac{s}{s^2+\omega^2}.\end{aligned}$$

Which holds for $s > 0$ as $|\cos\omega t| < 1$.

Notice, that differentiation is interchanged with multiplication by the Laplace transform (up addition of the value at the endpoint of course). Roughly, speaking this means that the "regularity" of a function can be seen in the growth rate of its Laplace transform.

The reverse is also true.

Proposition 5.9. *If $f(t)$ is piecewise linear and has Laplace transform $F(s)$ and $|f(t)| \leq Ce^{at}$ for some constant C, then for $s > a$*

$$\mathcal{L}\{tf(t)\}(s) = -\frac{d}{ds}F(s)$$

Proof.

$$\begin{aligned}F'(s) &= \frac{d}{ds}\int_0^\infty e^{-st}f(t)dt\\ &= \int_0^\infty \frac{\partial}{\partial s}\left(e^{-st}f(t)\right)dt\\ &= -\int_0^\infty te^{-st}f(t)dt\\ &= -\mathcal{L}\{tf(t)\}\end{aligned}$$

Of course we should justify interchanging the derivative and the improper integral in the second step – this is provided by the Leibniz integral rule. □

We remark that both of the preceding propositions can be iterated to give formulas for the nth derivative and multiplication by t^n.

Specifically, if $\mathcal{L}\{f^{(n)}(t)\} = F(s)$, then

$$\mathcal{L}\{f^{(n)}(t)\} = s^nF(s) - s^{n-1}f^{(n-1)}(0) + \ldots - f(0)$$

and

$$\mathcal{L}\{t^nf(t)\} = (-1)^nF^{(n)}(s).$$

Example 5.10. Let $F_n(s) = \mathcal{L}\{t^n\}$. We claim that

$$F_n(s) = \frac{n!}{s^{n+1}}$$

which holds for $s > 0$.

To see this note that our formula holds for $n = 0$. Assume it holds for $F_n(s)$. By the preceding Proposition,

$$F_{n+1}(s) = -\frac{d}{ds}F_n(s) = -\frac{d}{ds}(\frac{n!}{s^{n+1}}) = \frac{(n+1)!}{s^{n+2}}$$

and so the formula holds by induction. Here we used that $t^n \le Ce^{at}$ for some C (depeding on a) for all $a > 0$.

In general, one computes the Laplace transform of elementary functions by using the linearity and the two Propositions to reduce the function to a simple enough form that one can look up the answer using a table (such a table is at the end of these notes).

5.3. **The Laplace Transform of an ODE.** Lets apply the Laplace transform to investigate some simple constant coefficient inhomogenous ODEs.

Begin with the IVP

$$\begin{cases} y' = ay + g(t) \\ y(0) = y_0 \end{cases}$$

If $y(t)$ was the solution to this ODE, then the properties of the Laplace transform tell us that $Y(s) = \mathcal{L}\{y(t)\}$ would have to satisfy

$$\begin{aligned} sY(s) - y_0 &= \mathcal{L}\{y'(t)\} \\ &= \mathcal{L}\{ay + g(t)\} \\ &= aY(s) + G(s) \end{aligned}$$

where $G = \mathcal{L}\{g(t)\}$. Hence, we can solve for $Y(s)$ just using simple algebra. That is,

$$Y(s) = \frac{y_0 + G(s)}{s - a}.$$

Of course, we need to specify a domain on which Y is defined. In principle, this requires us to have informatoin about the solution $y(t)$. It turns out that it is enough to know $G(s)$, as one can show that the domain of $Y(s)$ will be the intersection of the domain of $G(s)$ and the interval (a, ∞).

Example 5.11. Consider the IVP

$$\begin{cases} y' = ay + \cos t \\ y(0) = y_0 \end{cases}$$

We have that

$$\begin{aligned} Y(s) &= \frac{y_0}{s-a} + \frac{s}{(s-a)(s^2+1)} \\ &= \frac{y_0}{s-a} + \frac{s}{(s-a)(s^2+1)} \\ &= \frac{y_0}{s-a} + \frac{1}{(a^2+1)(s^2+1)} - \frac{as}{(a^2+1)(s^2+1)} + \frac{a}{(a^2+1)(s-a)} \end{aligned}$$

where we used the partial fraction decomposition in the last step. This was important as we now can recognize $Y(s)$ as the Laplace transform of the function

$$y(t) = y_0 e^{at} + \frac{1}{a^2+1}\sin t - \frac{a}{a^2+1}\cos t + \frac{a}{a^2+1}e^{at}$$

which is readily checked to be the solution to the original IVP.

Similarly, for a second order ODE

$$\begin{cases} x'' + bx' + kx = g(t) \\ y(0) = x_0, y'(0) = x_1 \end{cases}$$

One obtains that the Laplace transform $X(s)$ of the solution $x(t)$ must satsify

$$s^2X(s) - sx_1 - x_0 + bsX(s) - bx_0 + kX(s) = G(s)$$

That is,

$$X(s) = \frac{sx_1 + (1+b)x_0}{s^2+bs+k} + \frac{G(s)}{s^2+bs+k}.$$

I leave it to you to work out the

Example 5.12. Consider the IVP

$$\begin{cases} x'' + x = 1 \\ x(0) = 0, x'(0) = 0 \end{cases}$$

If $X(s)$ is the Laplace transform of x, then

$$X(s) = \frac{1}{s(s^2+1)} = \frac{1}{s} - \frac{s}{s^2+1}$$

which we recognize as the Laplace trasform of the function

$$x(t) = 1 - \cos t$$

which is also the solution.

5.4. **The Inverse Laplace Transform.** While it is straightforward to solve for the Laplace transform of a solution to a constand coefficient ODE (at least assuming we can find the Laplace transform of the forcing), it is less straightforward to find the actual solution from its Laplace transform.

Let us denote by $\mathcal{L}^{-1}$, the *inverse Laplace transform* which is defined so that if $F(s) = \mathcal{L}\{f(t)\}$, then $f(t) = \mathcal{L}^{-1}\{F(s)\}$. To properly define the inverse Laplace transform (i.e., in terms of an integral) one needs to know some complex analysis (which is well beyond the scope of this course). Nevertheless, one can still compute the inverse Laplace transform using some of its properties (which are all easily derived from the corresponding properties for the actual Laplace transform)

Main properties

(1) Linearity:
$$\mathcal{L}^{-1}\{aF(s) + bG(s)\} = a\mathcal{L}^{-1}\{F(s)\} + b\mathcal{L}^{-1}\{G(s)\}$$
(2) Translation:
$$\mathcal{L}^{-1}\{F(s-a)\} = e^{at}\mathcal{L}^{-1}\{F(s)\}$$
(3) Scaling ($\omega > 0$):
$$\mathcal{L}^{-1}\{F(\omega s)\} = \omega^{-1}\mathcal{L}^{-1}\{F(s)\}(\omega^{-1}t)$$
(4) Differentiation
$$\mathcal{L}^{-1}\{F'(s)\} = -t\mathcal{L}^{-1}\{F\}$$

Unsurprisingly, one often encounters very complicated rational expressions, so a key technique in application is partial fraction decompostion.

Example 5.13. Compute

$$\mathcal{L}^{-1}\{\frac{7s+49}{s^2-3s-10}\}$$

Observe first that
$$s^2 - 3s - 10 = (s-5)(s+2)$$
Partial fractions implies that
$$\frac{7(s+7)}{s^2-3s-10} = \frac{A}{s-5} + \frac{B}{s+2} = \frac{12}{s-5} + \frac{-5}{s+2}$$
Hence,
$$\mathcal{L}^{-1}\{\frac{s+7}{s^2-3s-10}\} = 12\mathcal{L}^{-1}\{\frac{1}{s-5}\} - 5\mathcal{L}^{-1}\{\frac{1}{s+2}\}$$
$$= 12e^{5t} - 5e^{-2t}$$
where we used the translation property of the Laplace transform.

5.5. **Laplace Transform and Discontinuous Forcing.** The Laplace transform is particularly well suited to studying the problems with discontinuous forcing.

The key computation is that if $H_\tau(t)$ is the heaviside function that turns on at $t = a$, then
$$\mathcal{L}\{H_\tau(t)\} = \begin{cases} \frac{1}{s} & \tau \leq 0 \\ \frac{e^{-\tau s}}{s} & \tau > 0 \end{cases}$$
which holds for $s > 0$. Where we used that
$$\int_\tau^\infty e^{-st} dt = \frac{e^{-\tau s}}{s}.$$

Example 5.14. For $\omega > 0$, compute the Laplace transform of the solution to the IVP
$$\begin{cases} x'' + x = S_\omega(t) \\ x(0) = 0 \\ x'(0) = 0 \end{cases}$$
where
$$S_\omega(t) = +\sum_{n=0}^{\infty} \left(H_{2n\omega}(t) - H_{(2n+1)\omega}(t)\right)$$
is a square wave of period $2\omega > 0$.

To begin we compute the Laplace transform of $\mathcal{S}$. Using linearity, we have
$$\mathcal{L}\{S_\omega(t)\} = +\sum_{n=0}^{\infty} \left(\mathcal{L}\{H_{2n\omega}(t)\} - \mathcal{L}\{H_{(2n+1)\omega}(t)\}\right)$$
$$= \frac{1}{s}\sum_{n=0}^{\infty}\left(e^{-2n\omega s} - e^{-(2n+1)\omega s}\right)$$
$$= \frac{1}{s}\left(\frac{1-e^{-\omega s}}{1-e^{-2\omega s}}\right)$$
$$= \frac{1}{s}\left(\frac{1}{1+e^{-\omega s}}\right)$$
Technically, this needs to be justified (due to the infinite sum). However, this is straightforward as the sum is absolutely convergent. Hence, if $\mathcal{L}\{x(t)\} = X(s)$,

one has
$$s^2X(s) + X(s) = \mathcal{L}\{S_\omega(t)\} = \frac{1}{s}\left(\frac{1}{1+e^{-\omega s}}\right)$$
and so
$$X(s) = \frac{1}{s(e^{-\omega s}+1)(s^2+1)}.$$
Even without trying to compute the inverse laplace transform, there is an important distinction we can make between the case $\omega = (2n+1)\pi$ and $\omega \neq (2n+1)\pi$ for integer n. Namely, as
$$1 + e^{-\omega s}$$
has a zero at
$$s = \pm i\frac{(2n+1)\pi}{\omega}$$
for integer n, $X(s)$ has two double poles at $s = \pi i$ when $\omega = (2n+1)\pi$ but only simple poles when $\omega \neq (2n+1)\pi$. When $\omega = (2n+1)\pi$, then the period of S is $(4n+2)\pi$ which is exactly an integer multiple of the natural period of the homogenous system $x'' + x = 0$. As we will see, in this can be thought of in terms of resonance.

More generally, we have that for $\tau \geq 0$, and $f(t)$ any piecewise continuous function.
$$\mathcal{L}\{f(t-\tau)H_\tau(t)\} = e^{-\tau s}\mathcal{L}\{f(t)\}.$$
On your homework you will show that for piecewise continuous f, as long as $f(t) = 0$ for $t < 0$ and $\tau \geq 0$, then
$$\mathcal{L}\{f(t-\tau)\} = e^{-\tau s}\mathcal{L}\{f(t)\}$$
This directly implies the above fact, since
$$f(t-\tau)H_\tau(t) = g(t-\tau)$$
where $g(t) = f(t)H_0(t)$ and the inclusion of $H_0(t)$ doesn't matter in our definition of the Laplace transform. This formula is particularly useful for computing the inverse Laplace transform. Indeed,
$$\mathcal{L}^{-1}\{e^{-\tau s}F(s)\} = H_\tau(t)\mathcal{L}\{F(s)\}(t-\tau)$$
for $\tau \geq 0$.

One thing this should make clear (in case it wans't already) is that the Laplace transform solution method only solves forward in time (i.e. finds the solution for $t \geq 0$). Indeed, it doesn't see forcings for $t < 0$ at all and so can't possibly give the correct solution backwards in time.

Example 5.15. Use the Laplace transform to solve the IVP
$$\begin{cases} x'' + 4x = g(t) \\ x(0) = 0 \\ x'(0) = 0. \end{cases}$$
where
$$g(t) = \begin{cases} 0 & t < \pi \\ t - \pi & \pi \leq t \leq 2\pi \\ \pi & t \geq 2\pi \end{cases}$$

We write this as

$g(t) = (t-\pi)H_\pi(t) - (t-\pi)H_{2\pi}(t) + \pi H_{2\pi}(t) = (t-\pi)H_\pi(t) - (t-2\pi)H_{2\pi}(t)$

Hence,

$$G(s) = \mathcal{L}\{g(t)\} = \frac{e^{-\pi s}}{s^2} - \frac{e^{-2\pi s}}{s^2}$$

Hence, if $X(s) = \mathcal{L}\{x(t)\}$, then

$$\begin{aligned} X(s) &= \frac{e^{-\pi s} - e^{-2\pi s}}{s^2(s^2+4)} \\ &= \left(e^{-\pi s} - e^{-2\pi s}\right)\left(\frac{1/4}{s^2} - \frac{1/4}{s^2+4}\right) \end{aligned}$$

Hence,

$$\begin{aligned} x(t) &= \frac{1}{4}(t-\pi)H_\pi(t) - \frac{1}{8}(t-2\pi)H_{2\pi}(t) - \frac{1}{8}\sin 2(t-\pi)H_\pi(t) + \frac{1}{8}\sin 2(t-2\pi)H_{2\pi}(t) \\ &= \frac{1}{4}(t-\pi)H_\pi(t) - \frac{1}{4}(t-2\pi)H_{2\pi}(t) - \frac{1}{8}\sin 2tH_\pi(t) + \frac{1}{8}\sin 2tH_{2\pi}(t) \end{aligned}$$

Which was much simpler than other methods.

We conclude by noting that we can define the Laplace transform of the Dirac delta. Indeed, if $t_0 \neq 0$, then we have

$$\mathcal{L}\{\delta_{t_0}\} = \int_0^\infty e^{-st}\delta_{t_0}(t)dt = \begin{cases} 0 & t_0 < 0 \\ e^{-st_0} & t_0 > 0. \end{cases}$$

What happens at $t_0 = 0$? As usual this is slightly ambiguous. However, since the Laplace transform as we defined it is (as we pointed out earlier) suited only to solve forward in time (i.e. for $t > 0$). With this in mind, we define

$$\mathcal{L}\{\delta_0\} = \lim_{\tau\to 0^+} \mathcal{L}\{\delta_\tau\} = \lim_{\tau\to 0^+} e^{-s\tau} = 1.$$

One observation we make is that (for $t_0 \geq 0$).

$$\mathcal{L}\{H_{t_0}(t)\} = \frac{e^{-t_0 s}}{s}$$

And so (at least *formally*) when $t_0 > 0$,

$$\mathcal{L}\{\frac{d}{dt}H_{t_0}(t)\} = e^{-t_0 s} - H_{t_0}(0) = e^{-t_0 s} = \mathcal{L}\{\delta_{t_0}(t)\}$$

which is consistent with our previous notion as the Dirac delta as generalized derivative of the Heaviside function and justifies using the Laplace transform to solve ODEs with Dirac delta forcing. Again, as the Laplace transform considers only forward solutions, to make the formula work at $t_0 = 0$ one has

$$\mathcal{L}\{\frac{d}{dt}f(t)\} = s\mathcal{L}\{f(t)\} - \lim_{t\to 0^-} f(t)$$

which holds quite generally. Notice, this distinction only matters when f is not continuous at 0.

Example 5.16. For $\omega > 0$, find the Laplace transform of the solution to the IVP

$$\begin{cases} x'' + \omega^2 x = \sum_{n=0}^{\infty} \delta_{\pi n} \\ x(0) = 0 \\ x'(0) = 0. \end{cases}$$

This models an undamped harmonic oscillator that is hit periodically by a hammer.

Using the linearity of the Laplace transform we obtain that

$$\begin{aligned} \mathcal{L}\{\sum_{n=0}^{\infty} \delta_{\pi n}\} &= \sum_{n=0}^{\infty} \mathcal{L}\{\delta_{\pi n}\} \\ &= \sum_{n=0}^{\infty} e^{-n\pi s} \\ &= \frac{1}{1 - e^{-\pi s}} \end{aligned}$$

Hence,

$$X(s) = \frac{1}{1 - e^{-\pi s}(s^2 + \omega^2)}.$$

Again there is a distinction between $\omega = 2n$ and $\omega \neq 2n$ for integer n.

5.6. **The Laplace Tranform and Resonance.** Let us consider using the Laplace transform to solve the following IVP for $\omega > 0$

$$\begin{cases} x'' + x = \cos \omega t \\ x(0) = 0 \\ x'(0) = 0 \end{cases}$$

Computing the Laplace transform, $X(s)$, of a solution $x(t)$ we have that $X(s)$ satisfies

$$s^2 X(s) + X(s) = \frac{s}{s^2 + \omega^2}$$

That is,

$$X(s) = \frac{s}{(s^2 + 1)(s^2 + \omega^2)}$$

Notice, that we must distinguish between the case when $\omega = \pm 1$ and when $\omega \neq \pm 1$. In the latter case, the denominator of $X(s)$ has four distinct (complex) roots. This means that we can obtain a partial fraction decomposition

$$X(s) = \frac{1}{1 - \omega^2} \left(\frac{-s}{s^2 + 1} + \frac{s}{s^2 + \omega^2} \right)$$

Hence,

$$x(t) = \frac{1}{\omega^2 - 1} \left(\cos t - \cos \omega t \right).$$

However, when $\omega^2 = 1$ partial fractions clearly doesn't simplify anything. In this case, note that

$$X(s) = \frac{s}{(s^2 + \omega^2)^2} = -\frac{1}{2} \frac{d}{ds} \left(\frac{1}{s^2 + \omega^2} \right).$$

It is worth noting that this is the limit (by L'Hopitals rule) of the partial fraction decomposition as $\omega \to 1$.

Hence,

$$x(t) = \frac{1}{2} t \sin t.$$

The key takeaway is that we can detect a qualitative difference in the solutions (i.e. whether they are bounded or unbounded) in terms of the Laplace transform. This is significant, as often the hardest step in solving an IVP using the Laplace transform is taking the inverse Laplace transform (if one can do this in closed form at all).

To contextualize this, we note that for a constant coefficient ODE of order n,

$$x^{(n)} + a_{n-1}x^{(n-1)} + \ldots, +a_1 x' + a_0 = g(t)$$

there is a degree n polynomial associated to this ODE. Namely,

$$P(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \ldots + a_1\lambda + a_0$$

called the *characteristic polynomial.* You should check that this is actually the characteristic polynomial of the $n \times n$ matrix of the associated first order linear system. The point is that if $x(t)$ solves this ODE, then the Laplace transform, $X(s)$, of $x(t)$ satisfies

$$P(s)X(s) = H(s) + G(s)$$

where $H(s)$ is a degree $n-1$ polynomial that depends only on the initial conditions and $G(s)$ is the Laplace transform of $g(t)$. The natural frequencies of the system correspond to the zeros of $P(s)$. This is clearest when the zeros are purely imaginary. If a zero is $i\omega$, then their is a non-zero solution to the homogenous ODE which is $\frac{2\pi}{\omega}$ periodic and so its natural frequency is ω.

Example 5.17. The ODE

$$x'''' + 5x'' + 4 = 0$$

has characteristic polynomial

$$P(\lambda) = \lambda^4 + 5\lambda^2 + 4 = (\lambda^2 + 4)(\lambda^2 + 1)$$

hence there are solutions which are $2\pi/2 = \pi$ and $2\pi/1 = 2\pi$ periodic and the natural frequencies of the system are ± 2 and ± 1.

This is most easily seen by noting that the gnereal soluiton to this homogenous problem is

$$x(t) = c_1 \cos t + c_2 \sin t + c_3 \cos 2t + c_4 \sin 2t$$

Suppose now that $P(s)$ only has simple roots (i.e. there are no repeated roots). If any pole of $G(s)$ coincides with with a zero of $P(s)$, then $X(s)$ has a pole of higher order than that of $G(s)$, while if there is no overlap then all the poles of $X(s)$ are of the same order as $G(s)$ (along as some possible new ones – but these correspond to solutions of the homogenous problem).

Depending on the particular problem under consideration, either the solutions to the homogenous problem, or the forcing itself dominates or they are of the same "size"– we are usually interested in the latter two cases only. The correspond (as we will see below) to at least one pole of the $G(s)$ having real part greater than or equal to the largest real part of a zero of $P(s)$.

Furthermore, the qualitative behavior of solutions can (in most situations of interest) only change from the behavior of the dominant term when there is overlap, and this gives a straightforward way to find where resonance phenomena can occur.

When $G(s)$ is a rational function this is immediate from considerations involving partial fractions. However, we will give another approach to this in the next section.

5.7. **The Laplace Transform and the Qualitative Behavior of Solutions.** We conclude by discussing how to read off information about the long term behavior of a function from its Laplace transform. This is particularly important in using the Laplace transform to study solutions to ODEs as one is often only interested in the long term behavior of the solution and this perspective simplifies computations by allowing one to ignore a lot of terms.

The key idea is that the long term behavior is influenced only by the poles of the of Laplace transform. The real part of the pole corresponds to growth (if positive) or decay (if negative) as $t \to \infty$. The imaginary part corresponds to oscillatory behavior. Hence, the growth should be determined by the right most poles (i.e. those with largest real part).

While we've used the term pole informally before, its mathematical definition is:

Definition 5.18. A *pole* of a function F of a complex variable is just a point $a \in \mathbb{C}$, where F is singular but where $(s-a)^n F(s)$ is not singular for large enough $n \in \mathbb{N}$. The smallest such n is called the *order* of the pole. If the order is one, then we sometimes say we have a *simple* pole. A F has *pole at* $s = \infty$ if $\lim_{s\to\infty} F(s)$ does not exist (so F is singular at ∞), but $\lim_{s\to\infty} s^{-n}F(s)$ does.

Example 5.19. If
$$F(s) = \frac{P(s)}{Q(s)},$$
where $P(s)$ and $Q(s)$ are polynomials with no common zeros (so F is a *rational function* reduced to its smallest terms), then the poles of F exactly correspond to the zeros of $Q(s)$ and the order of the pole is exactly the order of the zero. Notice that rational functions also can have poles at ∞ (if the degree of Q is smaller than the degree of P) or no singularity (if the degree of P is less than or equal to the degree of Q).

When F is a rational function, poles are just points where F is singular. In general, while every pole is a point where F is singular the reverse need not be the case. While we will only discuss F whose singularities are poles in this class, its good to be able to recognize when this is not the case.

Example 5.20. On the one hand, if
$$F(s) = e^{-1/s}$$
then F is singular at $s = 0$, However, there is no n so that $s^n e^{-1/s}$ is non-singular. Hence, $s = 0$ is not a pole of F. However, $\lim_{s\to\infty} F(s) = 0$ and so F is not singular at $s = \infty$. On the other hand,
$$G(s) = e^s$$
has no poles for finite values of s. However, $\lim_{s\to\infty} s^{-n}e^s$ does not exist for any n and so G is singular at $s = \infty$ and does not have a pole there.

We now begin with a simple fact relating long-term behavior of a function with its Laplace transform. This is sometimes called the *Final Value Theorem.*

Theorem 5.21. *Suppose that $f(t)$ is differentiable and its Laplace transform $F(s) = \mathcal{L}\{f(t)\}$ is convergent for $s \in (0, \infty)$. If $\lim_{t\to\infty} f(t)$ exists, then*

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$$

Proof. We have

$$\begin{aligned}
sF(s) &= s\int_0^\infty e^{-st} f(t)dt \\
&= \lim_{k\to\infty} \int_0^k se^{-st} f(t)dt \\
&= \lim_{k\to\infty} \int_0^k -\frac{d}{dt}\left(e^{-st}\right) f(t)dt \\
&= \lim_{k\to\infty} \int_0^k -\frac{d}{dt}\left(e^{-st} f(t)\right) + e^{-st} f'(t)dt \\
&= \lim_{k\to\infty} \left(f(0) - e^{-sk} f(k) + \int_0^k e^{-st} f'(t)dt \right) \\
&= f(0) + \lim_{k\to\infty} \left(\int_0^k e^{-st} f'(t)dt \right)
\end{aligned}$$

Taking the limit as $s \to 0$ (to be completely accurate one must justify the interchange of limits, but we elide this point), gives

$$\lim_{s\to 0} sF(s) = \lim_{k\to\infty} \left(f(0) + \int_0^k f'(t)dt \right) = \lim_{k\to\infty} f(k)$$

Which completes the proof. □

Notice that to apply this theorem we need to know before hand that the limit actually exists. In many cases of interest, there will be a condition which we can verify about the Laplace transform which will tell us that this limit exists (and so what the value is):

Theorem 5.22. *Suppose that f is a piecewise continuous function with Laplace transform $F(s) = \mathcal{L}\{f(t)\}$ is convergent for $s \in (0, \infty)$. If all the singularities (read poles) of $sF(s)$ have negative real part, then*

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s).$$

The assumption on the singularities excludes functions which are singular at $s = \infty$.

Remark 5.23. Notice that we are only assuming that the improper integral defining the Laplace transform is converging for $s \in (0, \infty)$. Strictly speaking, this means that the Laplace transform is only defined for s with real part > 0. However, we are talking about F as if it is defined for almost every complex number. As we've seen with examples, we always get formula for F which make sense for almost all complex numbers, even when the integral doesn't converge. These formula for F are what we are using in the theorems. We again are not in a position to justify what we are doing. However, for future reference, the key word is analytic continuation.

This implies that the limit as $t \to \infty$ of $f(t)$ exists and is finite. In particular, f is bounded and has a definite limit as $t \to \infty$ (i.e., is not oscillatory like $\cos t$).

Remark 5.24. The condition that there is no pole at ∞ will prevent us from directly applying this result to some of the examples we considered previously – for instance the square wave. One can still study such functions, however greater care is needed and general theorems like the above are not as easy to state. As usual, a proper understanding of what is occuring in these cases is beyond the scope of this course.

Example 5.25. We check the theorems for $f(t) = e^{\alpha t}\cos t$. This has Laplace transform

$$F(s) = \mathcal{L}\{f(t)\} = \frac{s-\alpha}{(s-\alpha)^2+1}.$$

This function has a pole at $s = \alpha \pm i$ which has non-negative real part only when $\alpha \geq 0$. On the other hand, it's clear that $e^{\alpha t}\cos t$ has a limit as $t \to \infty$ only when $\alpha < 0$ and this limit is 0. However, $\lim_{s\to 0} sF(s) = 0$ for all values of α.

Hence, one must actually check the condition about the location of poles or know before hand that the $f(t)$ has a limit as $t \to \infty$. It is not enough to just look at the limit of $sF(s)$.

Notice, that when $\alpha = 0$, the functon f is bounded, but doesn't have a limit as the function $\cos t$ oscillates between -1 and 1.

More generally we have

Theorem 5.26. *Fix a $\alpha \in \mathbb{R}$. Suppose that $f(t)$ is differentiable and its Laplace transform $F(s) = \mathcal{L}\{f(t)\}$ is defined on (α, ∞). If $\lim_{t\to\infty} e^{-\alpha t} f(t)$ exists, then*

$$\lim_{t\to\infty} e^{-\alpha t} f(t) = \lim_{s\to\alpha} (s-\alpha)F(s)$$

Proof. Let $g(t) = e^{-\alpha t} f(t)$ and let $G(s)$ be the Laplace transform of g. As $\lim_{t\to\infty} g(t)$ exists, it is equal to

$$\lim_{s\to 0} sG(s) = \lim_{s\to 0} sF(s+\alpha) = \lim_{s\to\alpha} (s-\alpha)F(s).$$

This proves the claim. □

Theorem 5.27. *Fix $\alpha \in \mathbb{R}$. Suppose that f is a piecewise continuous function with Laplace transform $F(s) = \mathcal{L}\{f(t)\}$ defined on (α, ∞). suppose that $(s-\alpha)F(s)$ has no poles with real part greater than or equal to α, then*

$$\lim_{t\to\infty} e^{-\alpha t} f(t) = \lim_{s\to a} (s-\alpha)F(s).$$

That is, f grows like $e^{\alpha t}$.

Proof. Recall, that if $g(t) = e^{-\alpha t} f(t)$, then, $G(s)$, the Laplace transform of g satisfies

$$G(s) = F(s+\alpha)$$

Hence,

$$sG(s) = sF(s+\alpha)$$

has no poles with non-negative real part. Furthermore, $\lim_{s\to\alpha}(s-\alpha)F(s) = \lim_{s\to 0} sF(s+\alpha) = \lim_{s\to 0} sG(s)$. Hence, $g(t)$ satisfies $\lim_{t\to\infty} g(t) = \lim_{s\to\alpha}(s-\alpha)F(s)$ and so the claim follows from the special case discussed above. □

Example 5.28. Consider the third-order IVP

$$\begin{cases} x''' + x' = 1 + e^t \\ x(0) = 0, x'(0) = 0, x''(0) = 0 \end{cases}$$

First of all, we compute that $X(s)$, the Laplace transform of the solution satisfies

$$\begin{aligned} X(s) &= \frac{1}{s^3+s}\left(\frac{1}{s} + \frac{1}{s-1}\right) \\ &= \frac{1}{s^2(s^2+1)} + \frac{1}{s(s^2+1)(s-1)}. \end{aligned}$$

Where we used that the characteristic polynomial was

$$P(s) = s^3 + s.$$

By inspection this function has poles at $s = 0$ which is a double pole and a simple pole at $s = 1$ and possible simple poles at $s = \pm i$ (there might not be a pole due to cancellation – one has to actually do a computation). Notice, that there is no pole at $s = \infty$.

Since there is only one pole with real part 1 and this pole lies on the real axis, we conclude that $\lim_{t\to\infty} e^{-t} f(t) = C$ is finite. That is, the function grows no faster than $|C|e^t$ and at least this fast if $C \neq 0$. In fact, computing that

$$\lim_{s\to 1}(s-1)\left(\frac{1}{s^2(s^2+1)} + \frac{1}{s(s^2+1)(s-1)}\right) = \frac{1}{2}$$

we see that the function asympotically behaves like

$$\frac{1}{2}e^t + o(e^t)$$

as $t \to \infty$.

In general, we can get more refined results by expanding out the poles with largest real part. Specifically, suppose we fix an $\alpha \in \mathbb{R}$ and write $F(s)$ as

$$F(s) = \sum_{j=1}^{n} \frac{P_j(s-s_j)}{(s-s_j)^{m_j}} + F_0(s)$$

where here $m_j > 0$ are orders of poles at s_j, P_j are polynomials of order $k_j < m_j$ with $P_j(0) \neq 0$ and $(s-\alpha)F_0(s)$ has no poles real part $\geq \alpha$. We also assume that $s_j = \alpha_j + i\beta_j$ satisfies $\alpha_j \geq \alpha$, and so all the poles of the expansion are to the right of α and all the poles of F_0 are to the left.

Hence,

$$f(t) = \sum_{j=1}^{n} \mathcal{L}^{-1}\left\{\frac{P_j(s-s_j)}{(s-s_j)^{m_j}}\right\} + \mathcal{L}^{-1}\{F_0(s)\}$$

The point is that, on the one hand,

$$\mathcal{L}^{-1}\left\{\frac{P_j(s-s_j)}{(s-s_j)^{m_j}}\right\}$$

is something we can (more or less) easily look up on a table of Laplace transforms. In general it will a linear combination of terms like

$$C_j t^{l_j} e^{\alpha_j t} \cos \beta_j t + D_j t^{l_j} e^{\alpha_j t} \sin \beta_j t.$$

That is, all terms are of of growth bigger than $e^{\alpha t}$.

While on the other,

$$f_0(t) = \mathcal{L}^{-1}\{F_0\}$$

is, by the Theorem, smaller than $e^{\alpha t}$. In other words, the asymptotic behavior is dominated by the terms we have found in the expansion. Even when we can work out exactly what the solution is (for instance, for rational $F(s)$), if this is a very complicated function, we are still saving a lot of work.

Example 5.29. Consider again the third-order IVP

$$\begin{cases} x''' + x' = 1 + e^t \\ x(0) = 0, x'(0) = 0, x''(0) = 0 \end{cases}$$

Recall,

$$X(s) = \frac{1}{s^2(s^2+1)} + \frac{1}{s(s^2+1)(s-1)}$$

Which has a double pole at $s = 0$, a simple pole at $s = 1$ and possible simple poles at $s = \pm i$. Furthermore we saw that,

$$\lim_{s \to 1} (s-1)X(s) = \frac{1}{2}$$

As such, we can write

$$X(s) = \frac{1}{2}\frac{1}{s-1} + X_0(s)$$

where now $X_0(s)$ only has poles with real part that are less than or equal to zero.

As

$$\mathcal{L}^{-1}\left\{\frac{1}{2}\frac{1}{s-1}\right\} = \frac{1}{2}e^t,$$

we obtain that

$$X(t) = \frac{1}{2}e^t + x_0(t)$$

where

$$\lim_{t \to \infty} e^{-\alpha t} x_0(t) = 0$$

for all $\alpha > 0$, that is $x_0(t)$ grows slower than any exponentials (it could still grow like a polynomial however).

Example 5.30. Determine the asymptotics (as $t \to \infty$) of the solution to

$$\begin{cases} x'' + x = \sin t + e^{-2t} - 4e^{-4t} \\ x(0) = 0, x'(0) = 0, x''(0) = 0 \end{cases}$$

We compute that the Laplace transform $X(s)$ of $x(t)$ is of the form

$$X(s) = \frac{1}{(s^2+1)^2} + \frac{1}{(s^2+1)(s+2)} - \frac{4}{(s^2+1)(s+4)}$$

so this has double poles at $\pm i$ and simple poles at -2 and -4 (and no pole at ∞). This immediately implies that the solution grows slower than $e^{\alpha t}$ for any $\alpha > 0$.

Subtracting off the double pole at $\pm i$ we have that

$$\lim_{s\to i}(s^2+1)\left(X(s) - \frac{1}{(s^2+1)^2}\right) = \frac{1}{i+2} - \frac{4}{i+4} = \frac{2}{3} - \frac{i}{3} - \frac{16}{15} + \frac{4i}{15} = -\frac{6}{15} - \frac{1}{15}i$$

This means that

$$X(s) = \frac{1}{(s^2+1)^2} + \frac{-6-s}{15(s^2+1)} + X_0(s)$$

where $X_0(s)$ has no poles with part larger than -2. If you are having trouble seeing why this is, take the complex conjugate. Notice, we didn't need to do a full partial fractions expansion which saves a lot of effort.

Since,

$$\frac{1}{(s^2+1)^2} = \frac{1}{2}\frac{d}{ds}\left(\frac{s}{s^2+1}\right) + \frac{1}{2(s^2+1)}$$

we have

$$\mathcal{L}^{-1}\{\frac{1}{(s^2+1)^2}\} = -\frac{1}{2}t\cos t + \frac{1}{2}\sin t$$

Futhermore,

$$\mathcal{L}^{-1}\{\frac{-6-s}{15(s^2+1)}\} = -\frac{6}{15}\sin t - \frac{1}{15}\cos t$$

Hence,

$$x(t) = -\frac{1}{2}t\cos t + \frac{1}{2}\sin t - \frac{6}{15}\sin t - \frac{1}{15}\cos t + x_0(t)$$

where $x_0(t)$ decays like e^{-2t}. Indeed, one could work out

$$\lim_{t\to\infty} e^{2t}x_0(t)$$

if one wished.

In particular, $x(t)$ has at most linear growth.

Applying, these ideas to a special case of resonance we obtain the following (which is straightforward, if tedious, to generalize):

Theorem 5.31. *Suppose that $x(t)$ is a solution to the ODE*

$$x^{(n)} + a_{n-1}x^{(n-1)} + \dots, +a_1x' + a_0 = g(t).$$

where $g(t)$ is piecewise continuous. Let $G(s)$ be the Laplace transform of $g(t)$ and $P(\lambda)$ be characteristic polynomial of the ODE. Suppose there is an $\alpha \in \mathbb{R}$ so:

1. *$G(s)$ has either a single simple pole at $a_+ = a_- = \alpha$ or two simple poles at $a_\pm = \alpha \pm i\beta$, $\beta > 0$;*
2. *$G(s)$ has no poles with real part $> \alpha$ (this means no pole at ∞)*
3. *$P(\lambda)$ has only simple zeros none of which have real part greater than α.*

If the set of zeros of P does not contain $a_\pm$, then there is no resonance phenomena and

$$|x(t)| \leq Ce^{\alpha t}$$

for $C > 0$. If the set of zeros of P contain both $a_\pm$, then there is a resonance phenomena and

$$x(t) = Cte^{\alpha t}\cos\beta t + Dte^{\alpha t}\sin\beta t + x_0(t)$$

where $C^2 + D^2 > 0$ *and*

$$|x_0(t)| \leq Ce^{\alpha t}$$

as $t \to \infty$.

Notice this is most dramatic when $\alpha = 0$ as the distinction is then between bounded and unbounded (versus the distinction between $e^{\alpha t}$ and $te^{\alpha t}$).

6. Table of Laplace Transforms

Obtained from `http://planetmath.org/node/40588/source`.

Properties.

Original	Transformed	comment	derivation
$af(t) + bg(t)$	$a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}$	linearity	
$f(t) * g(t)$	$\mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\}$	convolution property	
$\int_a^b f(t,x)\,dx$	$\int_a^b \mathcal{L}\{f(t,x)\}\,dx$	integration with respect to a parameter	
$\frac{\partial}{\partial x} f(t,x)$	$\frac{\partial}{\partial x}\mathcal{L}\{f(t,x)\}$	diffentiation with respect to a parameter	
$f(\frac{t}{a})$	$aF(as)$	$\mathcal{L}\{f(t)\} = F(s)$	
$e^{at}f(t)$	$F(s-a)$	$\mathcal{L}\{f(t)\} = F(s)$	
$f(t-a)$	$e^{-as}F(s)$	$\mathcal{L}\{f(t)\} = F(s)$	
$t^n f(t)$	$(-1)^n F^{(n)}(s)$	$\mathcal{L}\{f(t)\} = F(s)$	
$\frac{f(t)}{t}$	$\int_s^\infty F(u)\,du$	$\mathcal{L}\{f(t)\} = F(s)$	
$\int_0^t f(u)\,du$	$\frac{F(s)}{s}$	$\mathcal{L}\{f(t)\} = F(s)$	
$f'(t)$	$sF(s) - \lim_{x\to 0+} f(x)$	$\mathcal{L}\{f(t)\} = F(s)$	
$f''(t)$	$s^2F(s) - s\lim_{x\to 0+} f(x) - \lim_{x\to 0+} f'(x)$	$\mathcal{L}\{f(t)\} = F(s)$	

Examples.

$f(t)$	$\mathcal{L}\{f(t)\}$	conditions	explanation	derivation
e^{at}	$\frac{1}{s-a}$	$s > a$		trivial
$\cos at$	$\frac{s}{s^2+a^2}$	$s > 0$		
$\sin at$	$\frac{a}{s^2+a^2}$	$s > 0$		
$\cosh at$	$\frac{s}{s^2-a^2}$	$s > \|a\|$		
$\sinh at$	$\frac{a}{s^2-a^2}$	$s > \|a\|$		
$\frac{\sin t}{t}$	$\arctan \frac{1}{s}$	$s > 0$	See sinc function	
t^r	$\frac{\Gamma(r+1)}{s^{r+1}}$	$r > -1, \quad s > 0$	gamma function Γ	
$e^{a^2 t} \operatorname{erf} a\sqrt{t}$	$\frac{a}{(s-a^2)\sqrt{s}}$	$s > a^2$	See error function	
$e^{a^2 t} \operatorname{erfc} a\sqrt{t}$	$\frac{1}{(a+\sqrt{s})\sqrt{s}}$	$s > 0$	See error function	
$\frac{1}{\sqrt{t}}$	$\sqrt{\frac{\pi}{s}}$	$s > 0$		
$J_0(at)$	$\frac{1}{\sqrt{s^2+a^2}}$	$s > 0$	Bessel function J_0	
e^{-t^2}	$\frac{\sqrt{\pi}}{2} e^{\frac{s^2}{4}} \operatorname{erfc}\left(\frac{s}{2}\right)$	$s > 0$	See error function	
$\ln t$	$-\frac{\gamma + \ln s}{s}$	$s > 0$	Euler'sconstant γ	
$\delta(t)$	1		Dirac delta function	

Rational Functions.

$f(t)$	$\mathcal{L}\{f(t)\}$	conditions	explanation	derivation
1	$\frac{1}{s}$			
t	$\frac{1}{s^2}$			
$\frac{t^{n-1}}{(n-1)!}$	$\frac{1}{s^n}$			
$\frac{1}{t+a}$	$e^{as}\mathrm{E}_1(as)$	$a > 0$	exponential integral E_1	
$\frac{1}{(t+a)^2}$	$\frac{1}{a} - se^{as}\mathrm{E}_1(as)$	$a > 0$		
$\frac{1}{(t+a)^n}$	$a^{1-n}e^{as}E_n(as)$	$a > 0, \quad n \in \mathbb{N}$	?	
$L_n(t)$	$\frac{1}{s}\left(\frac{s-1}{s}\right)^n$	$s > 0$	Laguerre polynomial L_n	