Practice Final Exam

1. Consider the following system

$$\begin{pmatrix} x_1 \\ x_1 \end{pmatrix}' = \begin{pmatrix} -4x_1(x_1^2-1)(x_2-2)^2 \\ -2x_2(x_1^2-1)^2 + 4(x_1-1)^2(x_1+1)^2 \end{pmatrix}$$

   (a) (10 points) Show this is a gradient system by determining the function V so that $\mathbf{X}' = -\nabla V$

   (b) (10 points) Determine all equilibria of the system and classify their type.

   (c) (10 points) For each sink determine the largest value $R > 0$ so that all points at distance less than R to the sink are in the basin of attraction of the sink. That is, if $\mathbf{X}_0$ is one such sink, then R is the largest value so that $\{\mathbf{X} : |\mathbf{X} - \mathbf{X}_0| < R\}$ is in the basin of attraction of $\mathbf{X}_0$.

2. (20 points) Give an example of an planar system for which the set $\Omega = \{\mathbf{X} : |\mathbf{X}| < 1\}$ is positively invariant and contains only one sink $\mathbf{X}_0$, but the basin of attraction of $\mathbf{X}_0$ is strictly small than Ω. (Recall, a region is *postiively invariant* if no solution with initial condition in Ω leaves Ω for $t \geq 0$).

3. Consider the planar system

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} x_1(2-x_2) \\ x_2(-2+4x_1) \end{pmatrix}$$

   (a) (10 points) Determine all equilibria of of the system and classify their type.

   (b) (10 points) Verify that the function

$$L(x_1, x_2) = 4x_1 - 2\log x_1 + x_2 - 2\log x_2$$

   which is defined for $x_1, x_2 > 0$ is a Lyapunov function for the system.

   (c) (10 points) Using L, what can you say about the stability of the equilibria?

4. Consider the forced 2×2 linear system

$$\mathbf{Y}' = A\mathbf{Y} + \begin{pmatrix} \cos t \\ e^t \end{pmatrix} \quad \text{where} \quad A = \begin{pmatrix} -2 & 1 \\ -9 & 4 \end{pmatrix}.$$

   (a) (10 points) Compute the matrix exponential e^{tA}.

   (b) (10 points) Find a particular solution to the equation (you may use any method you like).

5. (20 points) Determine a 3×3 linear system of ODEs which has the following properties: The phase portrait contains a stable line $x_1 = x_2 = x_3$ and the ellipse $x_1^2 + 4x_2^2 = 16$.

6. (30 points) Consider a mass on a spring whose motion is governed by

$$x'' + x = 0.$$

   Determine initial conditions x_0, v_0 (i.e., so $x(0) = x_0$ and $x'(0) = v_0$) which allow one to stop the mass completely after time $t = 2\pi$ by applying the impulse

$$I_{\lambda,\tau}(t) = \begin{cases} \lambda & \pi < t < \pi + \tau \\ 0 & \text{otherwise,} \end{cases}$$

   where $\lambda \geq 0$ and $0 \leq \tau \leq 2\pi$.

7. (20 points) Let

$$f(t) = \begin{cases} 2\cos 2t & t < 1 \\ 10t & 1 \leq t4 \\ e^{-t} & t \geq 4 \end{cases}$$

   and consider the second order equation

$$\begin{cases} x'' + 2x = f(t) \\ x(0) = 0, x'(0) = 1. \end{cases}$$

(a) (10 points) Determine $X(s)$, the Laplace transform of the solution.

(b) (10 points) Compute $\mathcal{L}^{-1}\{X(s)\}$. For what values of t does this give a valid solution?

8. (30 points) Consider the ODE

$$\begin{cases} x''' + 8x = g(t) \\ x(0) = 1, x'(0) = 0, x''(0) = 0. \end{cases}$$

(a) (10 points) Show that for $g(t) \equiv 0$, the solution is bounded.

(b) (10 points) Find a forcing $g(t)$ so that the solution $x(t)$ has the property $\lim_{t\to\infty} e^{-2t}x(t) = 1$.

(c) (10 points) Find a forcing so that $x(t)$ is unbounded, but, for t large, $|x(t)| \leq |t|$.