# Solutions Final Exam

1. (20 points) Determine a $3 \times 3$ linear system of ODEs whose phase portrait

   - contains an unstable line $x_1 = 2x_3, x_2 = 0$; and
   - contains the circle given by the intersection of the plane $x_1 + x_2 = 0$ with the unit sphere $x_1^2 + x_2^2 + x_3^2 = 1$.

   (Hint: Consider orthogonal unit length vectors in the plane $x_1 + x_2 = 0$.)

First of all we give a possible normal form for such a system. The given information tells us that

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

is such an example. Indeed, the corresponding system has the eigenvector $\mathbf{e}_1$ spanning an unstable line and the plane spanned by $\mathbf{e}_2$ and $\mathbf{e}_3$ contains circles.

We now need to find the linear transformation $T$ that transforms this matrix into the desired one, i.e. so $A = TBT^{-1}$. Observe that if $\mathbf{X}(t)$ solves

$$\mathbf{X}' = B \cdot \mathbf{X}$$

then $\mathbf{Y} = T \cdot \mathbf{X}$ solves

$$\mathbf{Y}' = (TBT^{-1}) \cdot \mathbf{Y} = A \cdot \mathbf{Y}.$$

Hence, if $T \cdot \mathbf{e}_1 = 2\mathbf{e}_1 + \mathbf{e}_3$, then $T \cdot \mathbf{e}_1$ spans the desired unstable line. Similarly, we observe that

$$\mathbf{e}_3$$

and

$$\frac{\sqrt{2}}{2}\mathbf{e}_1 - \frac{\sqrt{2}}{2}\mathbf{e}_2$$

span the plane $x_1 + x_2 = 0$, are unit length and are perpendicular. Hence, its enough to have $T \cdot \mathbf{e}_2 = \mathbf{e}_3$ and $T \cdot \mathbf{e}_3 = \frac{\sqrt{2}}{2}\mathbf{e}_1 - \frac{\sqrt{2}}{2}\mathbf{e}_2$. That is,

$$T = \begin{pmatrix} 2 & 0 & \frac{\sqrt{2}}{2} \\ 0 & 0 & -\frac{\sqrt{2}}{2} \\ 1 & 1 & 0 \end{pmatrix}$$

Hence,

$$A = TBT^{-1} = \begin{pmatrix} 2 & 0 & \frac{\sqrt{2}}{2} \\ 0 & 0 & -\frac{\sqrt{2}}{2} \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1/2 & 1/2 & 0 \\ -1/2 & -1/2 & 1 \\ 0 & -\sqrt{2} & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 - \frac{\sqrt{2}}{4} & 1 - \frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{2} \\ \frac{1}{2} & \frac{1}{2} + \sqrt{2} & 0 \end{pmatrix}$$

2. Consider the forced $2 \times 2$ linear system

$$\mathbf{Y}' = A\mathbf{Y} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \text{where} \quad A = \begin{pmatrix} 1 & -4 \\ 2 & -3 \end{pmatrix}.$$

(a) (10 points) Compute the matrix exponential $e^{tA}$.

The characteristic polynomial is

$$p_A(\lambda) = \lambda^2 + 2\lambda + 5$$

which has roots

$$\lambda = -1 \pm 2i.$$

The eigenvectors are found by solving

$$\begin{pmatrix} 2-2i & -4 \\ 2 & -2-2i \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = 0$$

that is

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} 1+i \\ 1 \end{pmatrix}$$

Hence,

$$T = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, T^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$$

and

$$T^{-1}AT = \begin{pmatrix} -1 & 2 \\ -2 & -1 \end{pmatrix}$$

Hence,

$$e^{At} = Te^{T^{-1}ATt}T^{-1} = Te^{-t} \begin{pmatrix} \cos 2t & \sin 2t \\ -\sin 2t & \cos 2t \end{pmatrix} T^{-1} = e^{-t} \begin{pmatrix} \cos 2t + \sin 2t & -2\sin 2t \\ \sin 2t & \cos 2t - \sin 2t \end{pmatrix}$$

(b) (10 points) Find a particular solution to the equation (you may use any method you like).

We guess that the solution is of the form

$$\mathbf{Y}_P(t) = \begin{pmatrix} a \\ b \end{pmatrix}$$

for $a$ and $b$ to be determined. For this to occur we must have

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} a - 4b \\ 2a - 3b \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Hence,

$$b = \frac{3}{5}, a = \frac{7}{5}.$$

and so

$$\mathbf{Y}_P(t) = \frac{1}{5}\begin{pmatrix} 7 \\ 3 \end{pmatrix}$$

3. (30 points) Consider a mass on a spring whose motion is governed by

$$x'' + 4x = 0.$$

Determine if it is possibly to completely stop the mass after time $t = 2\pi$ by applying the impulse

$$I_{\lambda,\tau}(t) = \begin{cases} \lambda & \pi < t < \pi + \tau \\ 0 & \text{otherwise,} \end{cases}$$

where $0 \le \lambda$ and $0 \le \tau < \pi$ when $x_0 \le 0$ and $v_0$ is arbitrary. Here $x_0 = x(0)$ and $v_0 = x'(0)$ are the initial conditions.

Set $y(t) = x(2\pi - t)$. We compute that $y$ satisfies the IVP

$$y'' + 4y = I_{\lambda,\tau}(2\pi - t) = \lambda H_{\pi-\tau}(t) - \lambda H_\pi(t)$$

where $H_a$ is the Heaviside function. We have that $y(2\pi) = x(0) = x_0$ and $y'(2\pi) = -x'(0) = -v_0$ and $y(0) = y'(0) = 0$. Computing the Laplace transform $Y(s)$ of $y$ we have

$$(s^2 + 4)Y(s) = \frac{\lambda e^{-(\pi-\tau)s}}{s} - \frac{\lambda e^{-\pi s}}{s}$$

Since

$$\frac{1}{(s^2+4)s} = \frac{1}{4}\left(\frac{s}{s^2+4} - \frac{1}{s}\right)$$

We have

$$Y(s) = \frac{\lambda}{4}\left(\frac{s(e^{-\pi s} - e^{-(\pi-\tau)s})}{s^2+4} - \frac{e^{-\pi s} - e^{-(\pi-\tau)s}}{s}\right)$$

and so

$$y(t) = \frac{\lambda}{4}\left(H_{\pi-\tau}(t) - H_\pi(t)\right) - \frac{\lambda}{4}\left(H_{\pi-\tau}(t)\cos 2(t - \pi + \tau) - H_\pi(t)\cos 2(t-\pi)\right)$$

For $t > \pi$ this gives

$$y(t) = \frac{\lambda}{4}\left(\cos 2(t-\pi) - \cos 2(t - \pi + \tau)\right)$$

and so

$$x_0 = \frac{\lambda}{4}\left(\cos 2\pi - \cos(2\pi + 2\tau)\right) = \frac{\lambda}{4}(1 - \cos 2\tau) = \frac{\lambda}{2}\sin^2\tau \ge 0$$

and likewise

$$v_0 = -\frac{\lambda}{4}\sin 2\tau = -\frac{\lambda}{2}\sin\tau\cos\tau$$

It is not possible to stop the mass unless $x_0 = 0$, since we are told $x_0 \le 0$ but must have $x_0 \ge 0$. Furthermore, if $x_0 = 0$, then either $\lambda = 0$ or $\sin\tau = 0$ and in either case we must also have $v_0 = 0$, i.e., it is only possible to stop the if $(x_0, v_0) = (0, 0)$.

4. Directly using the definition of the Laplace transform show that:

(a) (10 points) $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$ for $s > a$.

By definition

$$\begin{aligned}
\mathcal{L}\{e^{at}\}(s) &= \lim_{L\to\infty} \int_0^L e^{at} e^{-st} dt \\
&= \lim_{L\to\infty} \int_0^L e^{(a-s)t} dt \\
&= \lim_{L\to\infty} \frac{e^{(a-s)L} - 1}{a-s} \\
&= \frac{1}{s-a} + \frac{1}{a-s} \lim_{L\to\infty} e^{(a-s)L} \\
&= \frac{1}{s-a}
\end{aligned}$$

where the last equality holds because $a - s < 0$ (otherwise the limit is infinite).

(b) (10 points) $\mathcal{L}\{H_a(t)\} = \frac{e^{-as}}{s}$ for $s, a \geq 0$.

By definition

$$\begin{aligned}
\mathcal{L}\{H_a(t)\}(s) &= \lim_{L\to\infty} \int_0^L H_a(t) e^{-st} dt \\
&= \lim_{L\to\infty} \int_a^L e^{-st} dt \\
&= \lim_{L\to\infty} \frac{e^{-sL} - e^{-as}}{-s} \\
&= \frac{e^{-as}}{s} - \frac{1}{s} \lim_{L\to\infty} e^{-sL} \\
&= \frac{e^{-as}}{s}
\end{aligned}$$

here the second equaltity used that $a \geq 0$ and the last used that $s > 0$.

5. Consider the ODE

$$\begin{cases} x''' - x = g(t) \\ x(0) = a, x'(0) = b, x''(0) = c. \end{cases}$$

(a) (10 points) If $g(t) \equiv 0$, determine all the initial conditions $a, b, c$ so that the solution is bounded for $t \geq 0$.

Let $X(s) = \mathcal{L}\{x(t)\}$. We have that

$$(s^3 - 1)X(s) - as^2 - bs - c = 0$$

In order for the solution to be bounded for $t \geq 0$, we must have that $X(s)$ has no poles with real part that is positive. As

$$X(s) = \frac{as^2 + bs + c}{s^3 - 1} = \frac{as^2 + bs + c}{(s-1)(s^2 + s + 1)}$$

we see that this occurs if and only if $as^2 + bs + c$ has a root at $s = 1$. That is, $a + b + c = 0$.

(b) (10 points) Find a forcing $g(t)$ so that the solution $x(t)$ has the property $\lim_{t\to\infty} e^{-2t}x(t) = -2$.

As the zeros of the characteristic polynomial all have real part strictly less than 2, it suffices, by the Final Value Theorem, to find a function $g(t)$ which has Laplace transform with a simple pole at $s = 2$ An example of this is $g(t) = \alpha e^{2t}$ for $\alpha \neq 0$. Indeed, then

$$G(s) = \mathcal{L}\{g(t)\} = \frac{\alpha}{s-2}.$$

We allow $\alpha$ to be determined in order to get the correct limit. In this case, we have

$$X(s) = \frac{as^2 + bs + c}{s^3 - 1} + \frac{as^2 + bs + c}{s^3 - 1} + \alpha\frac{1}{(s-2)(s^3-1)}$$

The final value theorem implies that

$$\lim_{t\to\infty} e^{-2t}x(t) = \lim_{s\to 2}(s-2)X(s) = \frac{\alpha}{7}$$

Hence, $\alpha = -14$.

(c) (10 points) Assume the initial conditions satisfy $a = b = c = 0$, find a forcing so that $x(t)$ is unbounded, but, for $t$ large, $|x(t)| \leq C|t|$ for some $C > 0$.

Such a solution is consistent with resonance phenomena. In particular, if $g(t)$ is the forcing and $G(s) = \mathcal{L}\{g(t)\}$, then we have that

$$X(s) = \frac{G(s)}{s^3 - 1}$$

we need $G(s)$ to have double poles on the real axis and no other poles with real part greater than or equal to 0. For instance, the function

$$G(s) = \frac{(s-1)}{s^2}$$

satisfies these conditions. Hence,

$$g(t) = \mathcal{L}^{-1}\{\frac{s-1}{s^2}\} = 1 - t$$

is an example of the desired forcing.

6. Consider the planar system

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} (x_2^2 - x_2)(2x_2 - 1) \\ -x_1 \end{pmatrix}$$

(a) (10 points) Determine all equilibria of the system and classify their type (if possible) using the linearized equation.

Clearly the equalibria occur when $x_1 = 0$ and $x_2 = 0, \frac{1}{2}$ or 1. The linearized system at $(0,0)$ is

$$\mathbf{Y}' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cdot \mathbf{Y}$$

this has trace zero and determinant 1 so is not hyperbolic and so nothing can be said about the equilibria from this. The linearized system at $(0, \frac{1}{2})$ is

$$\mathbf{Y}' = \begin{pmatrix} 0 & -\frac{1}{2} \\ -1 & 0 \end{pmatrix} \cdot \mathbf{Y}$$

this has trace zero and determinant $-1/2$ and so is hyperbolic and is a saddle and so the equilibrium is a nonlinear saddle. The lineaized system at $(0,1)$ is

$$\mathbf{Y}' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cdot \mathbf{Y}$$

this has trace zero and determinant 1 so is not hyperbolic and so nothing can be said about the equilibria from this.

(b) (10 points) Verify that the function

$$L(x_1, x_2) = \frac{1}{2}\left(x_2^2(x_2-1)^2 + x_1^2\right)$$

is a Lyapunov function for the system.

We compute that

$$\begin{aligned}\dot{L}(x_1, x_2) &= \frac{\partial L}{\partial x_1}x_1' + \frac{\partial L}{\partial x_2}x_2' \\ &= x_1(x_2^2 - x_2)(2x_2 - 1) - (x_2^2 - x_2)(2x_2 - 1)x_1 \\ &= 0\end{aligned}$$

and so $L$ is a Lyapunov function for the system (but not a strict one). Indeed, the system is Hamiltonian with Hamiltonian $L$.

(c) (10 points) Using $L$, what more can you say about the stability of the equilibria?

We observe that $L$ has strict minima at $(0,0)$ and $(0,1)$ (this is becasue $L \geq 0$ and $L(0,0) = L(0,1) = 0$). Hence, since $L$ is a Lyapunov function that is never strict, $(0,0)$ and $(0,1)$ are stable equilibria and are not asymptotically stable.

7. (20 points) Give an example of an planar system for which the set $\Omega = \{\mathbf{X} : |\mathbf{X}| < 1\}$ is positively invariant and contains only one sink, $\mathbf{X}_0$, but the basin of attraction of $\mathbf{X}_0$ is strictly smaller than $\Omega$. (Recall, a region is *postively invariant* if no solution with initial condition in $\Omega$ leaves $\Omega$ for $t \geq 0$).

By the Poincare-Bendixson theorem, the only way this could occur is if there was a closed orbit in $\Omega$. The easiest way to construct such a solution rigorously is using polar coordinates. In polar coordinates, an example of the desired system would be of the form

$$\begin{cases} r' = -r(1-4r^2)^2 \\ \theta' = 1 \end{cases}$$

Notice, that for all $r \geq 0$, $r' \leq 0$ and so $\Omega$ is postively invariant. However, there is a closed orbit with $r = \frac{1}{2}$ and so the basin of attraction of $r = 0$ is contained inside of $\left\{r < \frac{1}{2}\right\}$. We can turn this into a system in cartesian coordinates by observing that $x_1 = r\cos\theta$ so

$$x_1' = r'\cos\theta - r\sin\theta\theta' = -x_1(1-4(x_1^2+x_2^2)^2 - x_2$$

and

$$x_2' = r'\sin\theta + r\cos\theta\theta' = -x_2(1-4(x_1^2+x_2^2))^2 + x_1.$$

8. Consider the following system

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} -4(x_1^2 + x_2^2 - 4)x_1 \\ -4(x_1^2 + x_2^2 - 4)x_2 - 12x_2 \end{pmatrix}$$

(a) (10 points) Show this is a gradient system by determining the function $V$ so that $\mathbf{X}' = -\nabla V$

Suppose that $V$ was such a function. Then we would have

$$\frac{d}{dx_1}V = 4(x_1^2 + x_2^2 - 4)x_1$$

Integrating we see that

$$V = x_1^4 + 2x_2^2x_1^2 - 8x_1^2 + h(x_2)$$

Hence,

$$4(x_1^2 + x_2^2 - 4)x_2 + 12x_2 = \frac{d}{dx_2}V = 4x_2x_1^2 + h'(x_2)$$

and so

$$h'(x_2) = 4x_2^3 - 16x_2 + 12x_2$$

that is

$$h(x_2) = x_2^4 - 8x_2^2 + 6x_2^2$$

so

$$V(x_1, x_2) = x_1^4 + 2x_1^2x_2^2 + x_2^4 - 8(x_1^2 + x_2^2) + 6x_2^2 = (x_1^2 + x_2^2 - 4)^2 - 16 + 6x_2^2$$

(b) (10 points) Determine all equilibria of the system and classify their type (if possible) using the linearized equation. (Hint: there are five in total).

For an equilibrium to appear we first must first have $-4(x_1^2 + x_2^2 - 4)x_1 = 0$. This can only happen if $x_1 = 0$ or if $x_1^2 + x_2^2 = 4$. We must also have $-4(x_1^2 + x_2^2 - 4)x_2 - 12x_2 = 0$. In the case that $x_1 = 0$, this reduces to $-4x_2^3 + 4x_2 = -4x_2(x_2^2 - 1)$. In particular, we have equilibria at $(0, 0)$ and $(0, \pm 1)$. In the case that, $x_1^2 + x_2^2 = 4$ we have that $x_2 = 0$ and so $x_1 = \pm 2$ i.e. $(\pm 2, 0)$ are also equlibria.

If $(x_1, x_2)$ is an equlibria we compute that the variational equation is

$$\mathbf{Y}' = \begin{pmatrix} -4(x_1^2 + x_2^2 - 4) - 8x_1^2 & -8x_1x_2 \\ -8x_1x_2 & -4(x_1^2 + x_2^2 - 4) - 8x_2^2 - 12 \end{pmatrix} \cdot \mathbf{Y}$$

At $(0, 0)$ we obtain the linearized matrix

$$\begin{pmatrix} 16 & 0 \\ 0 & 4 \end{pmatrix}$$

which has two positive eigenvalues so is a source. At $(0, \pm 1)$ we obtain the linearized matrix

$$\begin{pmatrix} 12 & 0 \\ 0 & -8 \end{pmatrix}$$

which has a positive and a negative eigenvalue and so is hyperbolic and is a saddle. At $(\pm 2, 0)$ the linearized matrix is

$$\begin{pmatrix} -32 & 0 \\ 0 & -12 \end{pmatrix}$$

this has strictly negative eigenvalues and so both equlibria are sinks.

(c) (10 points) Using the $V$ you found in part a), determine the largest value $V_0$ so that all points, $\mathbf{X}$, with $V(\mathbf{X}) < V_0$ are in the basin of attraction of a sink of the system.

Observe first that $V$ is a strict Lyapunov function for the system away from the critical points of $V$. We have that at the two sinks $(\pm 2, 0)$ that $V = -16$ (note that we could have always added a constant when finding $V$ so this value is arbitrary). While at the saddle equlibria $(0, \pm 1)$ we have $V = -1$ and at the source $(0, 0)$ we have $V = 0$. Hence, as long as $-16 < V(\mathbf{X}) < -1$ we have that $V$ strictly decreases along a solution to the system with initial value $\mathbf{X}$ and so the basin of attraction of the sinks is contained with in the set $\{\mathbf{X} : V(\mathbf{X}) < -1\}$.