Practice Midterm Exam 1

1. Find *all* solutions $y = y(x)$ to the following initial value problems (remember to include domain):
   (a) (10 points)
$$\begin{cases} y' = (1+y^2)x \\ y(0) = 1 \end{cases}$$
   (b) (10 points)
$$\begin{cases} y' = y^{2/3} \\ y(0) = 0 \end{cases}$$

2. Put the following matrices in canonical form (i.e., Jordan normal form).
   (a) (10 points)
$$A_1 = \begin{pmatrix} -1 & 1 \\ -9 & 5 \end{pmatrix}$$
   (b) (10 points)
$$A_2 = \begin{pmatrix} 1 & 2 \\ -4 & -3 \end{pmatrix}$$

3. Determine a 2×2 linear system of ODEs which has the following properties:
   (a) (10 points) The phase portrait contains a stable line $y = 3x$ and an unstable line $x = 0$.
   (b) (10 points) The phase portrait contains the ellipse $5x^2 - 4xy + y^2 = 20$.

4. (20 points) Find the general solution to the the following 3×3 linear system:
$$\mathbf{Y}' = \begin{pmatrix} 7 & 0 & -3 \\ -9 & -2 & 3 \\ 18 & 0 & -8 \end{pmatrix} \mathbf{Y}$$

5. Consider the following one-parameter family of autonomous ODEs
$$y' = F_a(y) = \frac{y}{1+y^2} - ay$$
   (a) (10 points) Draw the bifurcation diagram for this family of ODEs.
   (b) (10 points) Show that there is a value a_0 so that if $a_- < a_0$ and $a_+ > a_0$, then the systems $y' = F_{a_-}(y)$ and $y' = F_{a_+}(y)$ are *not* topologically conjugate (Hint: Do not try and solve the ODEs explicitly).