# Solutions Midterm Exam 1

1. Find *all* solutions $y = y(x)$ to the following initial value problems (remember to include domain):

   (a) (10 points)

$$\begin{cases} x^2y' = y(1+y^2) \\ y(0) = 0 \end{cases}$$

(Hint: use that $\frac{1}{y(1+y^2)} = \frac{1}{y} - \frac{y}{y^2+1}$).

We first observe that this ODE is poorly behaved when $x = 0$ and so we should take care. With that in mind, we separate variables and see that any non-zero solution must satisfy (away from $x = 0$)

$$\frac{y'}{y(1+y^2)} = \frac{1}{x^2}$$

That is (by some calculus),

$$\frac{d}{dy}\left(\ln|y| - \frac{1}{2}\ln(1+y^2)\right) = \frac{d}{dx}\left(-\frac{1}{x} + C_0\right)$$

That is,

$$\frac{|y|}{\sqrt{1+y^2}} = e^{-1/x+C_0}$$

which we recognize as

$$\frac{y}{\sqrt{1+y^2}} = Ce^{-1/x}$$

Solving for $y$, gives

$$y = \frac{Ce^{-1/x}}{\sqrt{1-C^2e^{-2/x}}}.$$

We observe that on the one hand, if $C = 0$, they $y(x) = 0$ for all $x$ and so is a global solution. On the other hand, if $C \neq 0$, then this function is not continuous at 0. However, the piecewise, function

$$y(x) = \begin{cases} 0 & x \leq 0 \\ \frac{Ce^{-1/x}}{\sqrt{1-C^2e^{-2/x}}} & x > 0 \end{cases}$$

is continuous (even differentiable) at $x = 0$ and solves the IVP. However, this solution has another singularity if it ever happens that

$$1 - C^2e^{-2/x} = 0$$

which occurs when $x = \frac{1}{\ln|C|}$ Notice, this is positive real only when $|C| > 1$ i.e. there is no singularity for $-1 \leq C \leq 1$. Putting this all together, we obtain that all solutions are of the form

$$y(x) = \begin{cases} 0 & x \leq 0 \\ \frac{Ce^{-1/x}}{\sqrt{1-C^2e^{-2/x}}} & x > 0 \end{cases}$$

which has domain $(-\infty, \infty)$ for $-1 \leq C \leq 1$ and $(-\infty, \frac{1}{\ln|C|})$ for $|C| > 1$.

(b) (10 points)

$$\begin{cases} y' = |y|^{3/2} \\ y(0) = 1 \end{cases}$$

Separating variables, we obtain

$$\frac{y'}{|y|^{3/2}} = \frac{d}{dy}\left(-2|y|^{-1/2}\right) = \frac{d}{dx}\left(x + C_0\right)$$

That is,

$$|y| = \frac{4}{(x + C_0)^2}$$

note that (by inspection)

$$y = \frac{4}{(x + C_0)^2} > 0$$

is only a solution for $x < -C_0$ while

$$y = -\frac{4}{(x + C_0)^2} < 0$$

is only a solutions for $x > -C_0$. There is also the global solution $y(x) = 0$ (corresponding to $C_0 = \infty$). Hence, to satisfy the initial condition we must have $C_0 < 0$. In this case, one checks that $C_0 = -2$. Hence, the solution is given by

$$y(x) = \frac{4}{(x - 2)^2}$$

which has domain $(-\infty, 2)$. Note, that each choice of initial condiiton gives a unique choice of $C_0$, so this is the unique solution.

2. Put the following matrices in canonical form (i.e., Jordan normal form). That is, find an invertible matrix $T$ so that $T^{-1}AT$ is in canonical form (and determine this form).

   (a) (10 points)

$$A_1 = \begin{pmatrix} 2 & 9 \\ -1 & -4 \end{pmatrix}$$

The characteristic polynomial is $\lambda^2 + 2 + 1 = (\lambda + 1)^2$ and so their is one eigenvalue with algebraic multiplicity 2. Since, the matrix is not diagonal, this eigenvalue has geometric multiplicity 1. Solving the system

$$(A_1 - (-1)I)\,\mathbf{v} = 0$$

one finds that

$$\mathbf{v} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$$

is an eigenvector. Any other vector which is linearly independent will work as a generalized eigenvector i.e., a solution to

$$(A_1 - (-1)I)\,\mathbf{w} = \alpha\mathbf{v}$$

for $\alpha \neq 0$ and so we choose

$$\mathbf{w} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Notice that

$$(A_1 - (-1)I)\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \end{pmatrix} = \mathbf{v}$$

so in this case $\alpha = 1$ (otherwise we would multiply $\mathbf{w}$ by $\alpha^{-1}$). Hence,

$$T = \begin{pmatrix} \mathbf{v} & \mathbf{w} \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ -1 & 0 \end{pmatrix}$$

is invertible and puts $A_1$ in canonical form, i.e.,

$$T^{-1}A_1T = \begin{pmatrix} -1 & \alpha \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}.$$

Note that $T$ is not unique.

(b) (10 points)

$$A_2 = \begin{pmatrix} 5 & 10 \\ -1 & -1 \end{pmatrix}$$

The characteristic polynomial is $\lambda^2 - 4\lambda + 5 = (\lambda - 2)^2 + 1$ and so $A_2$ has complex eigenvalues $2 \pm i$. One finds a complex eigenvector by solving

$$(A_2 - (2+i)I)\,\mathbf{z} = 0$$

which gives

$$\mathbf{z} = \begin{pmatrix} 1 - 3i \\ i \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + i \begin{pmatrix} -3 \\ 1 \end{pmatrix} = \mathbf{x} + i\mathbf{y}.$$

Hence, setting

$$T = \begin{pmatrix} \mathbf{x} & \mathbf{y} \end{pmatrix} = \begin{pmatrix} 1 & -3 \\ 0 & 1 \end{pmatrix}$$

gives

$$T^{-1} A_2 T = \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}$$

which is the canonical form.

3. Determine a $2 \times 2$ linear system of ODEs which has the following properties:

(a) (10 points) The phase portrait contains a stable line $y = 2x$ and no other stable or unstable lines.

The desired matrix $A$ must have repeated negative eigenvalue (as this is the only way to have a single stable line). As such, we may take its canonical form to be

$$\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$$

which has as its unstable line the $x$-axis. To finish the problem, we need to find a linear transformation $T$ so that

$$T^{-1}AT = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$$

To do so, we observe that $A$ must have its sole eigenvector on the line $y = 2x$ – i.e. be of the form $\mathbf{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ which gives the first column of $A$. The generalized eigenvector can be any vector linearly independent of $\mathbf{v}$, for instance $\mathbf{e}_2$, so we may take

$$T = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$$

and

$$A = T\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}T^{-1} = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} -3 & 1 \\ -4 & 1 \end{pmatrix}.$$

(b) (10 points) The phase portrait contains the ellipse $x^2 + 4y^2 = 16$.

The desired matrix $A$ must purely imaginary eigenvalues. We take $\pm i$. Hence, its canonical form is

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The phase portrait of this matrix consists of concentric circles. We now need to find the matrix $T$. To do so it suffices to observe that the matrix $T$ takes the phase diagram of the canonical system to the phase diagram of $A$. Since the ellipse $x^2 + 4y^2 = 16 = x^2 + (2y)^2 = 16$ is obtained from the circle $x^2 + y^2 = 16$ by sending $(x, y) \mapsto (x, y/2)$ we may take

$$T = \begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix}$$

so

$$A = T\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}T^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ -1/2 & 0 \end{pmatrix}$$

4. Consider

$$A = \begin{pmatrix} 1 & 1 & 2 \\ -5 & -5 & -2 \\ 1 & -1 & 0 \end{pmatrix}$$

(a) (10 points) Find the general solution to the the $3 \times 3$ linear system:

$$\mathbf{Y}' = A\mathbf{Y}$$

The characteristic polynomial is $-\lambda^3 - 4\lambda^2 + 4\lambda + 16 = -(\lambda - 2)(\lambda + 2)(\lambda + 4)$. A remark on how to factor this – if the roots are integers (which is likely for an exam problem) then they must divide the constant term. Hence, the eigenvalues of the matrix are $-4, -2, 2$ with (after some computation) corresponding eigenvectors

$$\begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

so

$$T = \begin{pmatrix} -1 & -1 & 1 \\ 3 & 1 & -1 \\ 1 & 1 & 1 \end{pmatrix}$$

is invertible and

$$T^{-1}AT = \begin{pmatrix} -4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

Hence, the general solution is

$$\mathbf{Y}(t) = T \begin{pmatrix} e^{-4t} & 0 & 0 \\ 0 & e^{-2t} & 0 \\ 0 & 0 & e^{2t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} -1 & -1 & 1 \\ 3 & 1 & -1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} e^{-4t} & 0 & 0 \\ 0 & e^{-2t} & 0 \\ 0 & 0 & e^{2t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

$$= c_1 e^{-4t} \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix} + c_2 e^{-2t} \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} + c_3 e^{2t} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

(b) (10 points) Solve the inital value problem

$$\begin{cases} \mathbf{Y}' = AY \\ \mathbf{Y}(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \end{cases}$$

By the above

$$\mathbf{Y}(0) = T \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

so (two steps of Gaussian elimination reduces $T$ to upper triangular form)

$$\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = T^{-1} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1/2 \\ -1 \\ 1/2 \end{pmatrix}$$

5. Consider the following one-parameter family of autonomous ODEs

$$y' = F_a(y) = \frac{a}{1+y^2} + y^2$$

(a) (10 points) Draw the bifurcation diagram for this family of ODEs.

One first finds the equilibria of the family of ODEs. That is determines when

$$0 = F_a(y) = \frac{a}{1+y^2} + y^2$$

since $1+y^2$ is never zero this is equivalent to solving

$$0 = y^4 + y^2 + a = 0 = (y^2 + \frac{1}{2})^2 + a - \frac{1}{4}.$$

In particular, there are no solutions if $a > \frac{1}{4}$ on the other hand, if $a \leq \frac{1}{4}$ then $y^2 = \sqrt{\frac{1}{4} - a} - \frac{1}{2}$ (note taking negative square root is always negative so can't have a solution). Notice, that this has no solutions unless $\frac{1}{4} - a \geq \frac{1}{4}$. That is, $a \leq 0 < \frac{1}{4}$. Once, $a \leq 0$ there are two solutions of opposite sign. So the equilibria look sort of like a parabola on its side opening in the negative $a$ direction. As $F_a(0) > 0$ for $a > 0$, we see that all solutions increase in this range (so the arrows are up) while for $a < 0$, one has $F_a(0) < 0$ while for $y$ with $|y|$ large, $F_a(y) > 0$ so the arrows are down inside the parabola and up outside of it.

(b) (10 points) Show that there is a value $a_0$ so that if $a_- < a_0$ and $a_+ > a_0$, then the systems $y' = F_{a_-}(y)$ and $y' = F_{a_+}(y)$ are *not* topologically conjugate (Hint: Do not try and solve the ODEs explicitly).

The value $a_0 = 0$ is such a number. Indeed, let $\phi_t^{a-}$ and $\phi_t^{a+}$ be the corresponding flows. If these were conjugate there would be a homoemorphism (i.e 1-1, onto and continuous with continuous inverse) so that $\phi_t^{a+}(h(y_0)) = h(\phi_t^{a-}(y_0))$ for all $y_0$. Letting $y_0$ be one of the equilibria of $y' = F_{a_-}(y)$, we have that $\lim_{t\to\infty} h(\phi_t^{a-}(y_0)) = h(y_0)$ a finite number but $\lim_{t\infty}(\phi_t^{a+}(h(y_0))) = \infty$ so the two can't be conjugate.