Practice Midterm Exam 2

1. Consider the forced 2×2 linear system

$$\mathbf{Y}' = A\mathbf{Y} + \begin{pmatrix} te^t \\ 0 \end{pmatrix} \quad \text{where} \quad A = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}.$$

   (a) (10 points) Compute the matrix exponential e^{tA}.

   (b) (10 points) Find a particular solution to the equation (you may use any method you like).

2. (20 points) Consider a mass on a spring whose motion is governed by

$$x'' + x = 0.$$

   Determine initial conditions x_0, v_0 (i.e., so $x(0) = x_0$ and $x'(0) = v_0$) which allow one to stop the mass completely after time $t = \pi$ by a single blow with a hammer (transmitting any force a) at time $t = \pi$.

3. (20 points) Let $H_a(t)$ is the heaviside function which "turns on" at $t = a$. For $a > 0$, consider the IVP

$$\begin{cases} x'' - x = (t-a)H_a(t) + 1 - H_{-1}(t) \\ x(0) = 1, x'(0) = 0. \end{cases}$$

   (a) (10 points) Determine $X(s)$, the Laplace transform of the solution.

   (b) (10 points) Compute $\mathcal{L}^{-1}\{X(s)\}$. For what values of t does this give a valid solution?

4. (20 points) Consider the ODE

$$\begin{cases} x''' + x = g(t) \\ x(0) = 1, x'(0) = 0, x''(0) = 0. \end{cases}$$

   Determine a forcing $g(t)$ so that the solution $x(t)$ has the property that it grows larger, as $t \to \infty$, than any solution to the homogenous problem and larger than $g(t)$.

5. Consider the autonomous non-linear ODE

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} x_2 \sin x_1 \\ x_1 - x_2 \end{pmatrix}.$$

   (a) (15 points) For any $\alpha \in \mathbb{R}$ consider the initial conditions:

$$\begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} \pi \\ \alpha \end{pmatrix}.$$

   Compute the the Picard iterates, $\mathbf{U}_k$, for these solutions and verify that they converge to a global solution to the IVP.

   (b) (5 points) For each β, determine the variational equation along the solutions

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} \pi \\ \beta e^{-t} + \pi \end{pmatrix}.$$