Solutions Midterm Exam 2

1. Consider the forced 2×2 linear system

$$\mathbf{Y}' = A\mathbf{Y} + \begin{pmatrix} -3e^t \\ 5e^t \end{pmatrix} \quad \text{where} \quad A = \begin{pmatrix} -14 & -9 \\ 25 & 16 \end{pmatrix}.$$

(a) (10 points) Compute the matrix exponential e^{tA}.

The characteristic polynomial of A is $p_A(\lambda) = \lambda^2 - 2\lambda + 1$. Hence, the there is one eigenvalue $\lambda = 1$ with multiplicity 2. The space of solutions to

$$(A - I)\mathbf{v} = \begin{pmatrix} -15 & -9 \\ 25 & 15 \end{pmatrix} \mathbf{v}$$

is spanned by the eigenvector

$$\mathbf{v} = \begin{pmatrix} -3 \\ 5 \end{pmatrix}.$$

Similarly, a generalized eigenvector is given by

$$\mathbf{w} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}.$$

Hence, setting

$$T = \begin{pmatrix} -3 & -1 \\ 5 & 2 \end{pmatrix}$$

we have

$$A = T \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} T^{-1}$$

and so

$$e^{tA} = T \begin{pmatrix} e^t & te^t \\ 0 & e^t \end{pmatrix} T^{-1} = \begin{pmatrix} -3 & -1 \\ 5 & 2 \end{pmatrix} \begin{pmatrix} e^t & te^t \\ 0 & e^t \end{pmatrix} \begin{pmatrix} -2 & -1 \\ 5 & 3 \end{pmatrix} = e^t \begin{pmatrix} 1 - 15t & -9t \\ 25t & 1 + 15t \end{pmatrix}$$

(b) (10 points) Find a particular solution to the equation (you may use any method you like).

We use the variation of parameters formula to see that a particular solution is given by

$$\begin{aligned}
\mathbf{Y}_p &= \int_0^t e^{(t-s)A} \begin{pmatrix} -3e^s \\ 5e^s \end{pmatrix} ds \\
&= e^{tA} \int_0^t e^{-s} \begin{pmatrix} 1 + 15s & 9s \\ -25s & 1 - 15s \end{pmatrix} \begin{pmatrix} -3e^s \\ 5e^s \end{pmatrix} ds \\
&= e^{tA} \int_0^t \begin{pmatrix} -3 \\ 5 \end{pmatrix} ds \\
&= e^t \begin{pmatrix} 1 - 15t & -9t \\ 25t & 1 + 15t \end{pmatrix} \begin{pmatrix} -3t \\ 5t \end{pmatrix} \\
&= te^t \begin{pmatrix} -3 \\ 5 \end{pmatrix}
\end{aligned}$$

2. (20 points) Consider a mass on a spring whose motion is governed by

$$x'' + x = 0.$$

Determine initial conditions x_0, v_0 (i.e., so $x(0) = x_0$ and $x'(0) = v_0$) which allow one to stop the mass completely after time $t = \frac{\pi}{4}$ by a single blow with a hammer (transmitting any force a) at time $t = \frac{\pi}{4}$.

We turn this into a backwards in time problem. Specifically it is equivalent to solving the IVP

$$\begin{cases} x'' + x = 0 \\ x\left(\frac{\pi}{4}\right) = 0, x'\left(\frac{\pi}{4}\right) = -a. \end{cases}$$

and determining $x(0)$ and $x'(0)$. In any event, a general solution to this problem is given by $x(t) = c_1 \cos t + c_2 \sin t$

The initial conditions imply

$$\frac{\sqrt{2}}{2} c_1 + \frac{\sqrt{2}}{2} c_2 = 0$$

and

$$-\frac{\sqrt{2}}{2} c_1 + \frac{\sqrt{2}}{2} c_2 = -a$$

so

$$x(t) = \frac{\sqrt{2}}{2} a \cos t - \frac{\sqrt{2}}{2} a \sin t$$

Hence,

$$x_0(0) = \frac{\sqrt{2}}{2} a$$

and

$$v_0 = x'(0) = -\frac{\sqrt{2}}{2} a$$

where $a \in \mathbb{R}$.

3. Let $H_a(t)$ is the heaviside function which "turns on" at $t = a$. For $a > 0$, consider the IVP

$$\begin{cases} x'' + x = e^t H_a(t) + H_{-3}(t) \\ x(0) = 0, x'(0) = 1. \end{cases}$$

(a) (10 points) Determine $X(s)$, the Laplace transform of the solution.

Writing the ODE as

$$x'' + x = e^a e^{t-a} H_a(t) + H_{-3}(t)$$

Using the table of Laplace transforms one computes that the Laplace transform satisfies

$$s^2 X(s) - 1 + X(s) = \frac{e^{a-as}}{s-1} + \frac{1}{s}$$

where we used that $H_{-3}(t) = 1$ for $t \geq 0$. Hence,

$$X(s) = \frac{e^{a-as}}{(s-1)(s^2+1)} + \frac{1}{s(s^2+1)} + \frac{1}{s^2+1}$$

(b) (10 points) Compute $\mathcal{L}^{-1}\{X(s)\}$. For what values of t does this give a valid solution?

In order to compute the inverse Laplace transform, we use partial fractions to write

$$\frac{1}{s(s^2+1)} = \frac{1}{s} - \frac{s}{s^2+1}$$

and

$$\frac{1}{(s-1)(s^2+1)} = \frac{1}{2(s-1)} - \frac{s+1}{2(s^2+1)}$$

Hence, the inverse Laplace transform is

$$\mathcal{L}^{-1}\{X(s)\} = \frac{e^a}{2} e^{(t-a)} H_a(t) - \frac{e^a}{2} \left(\cos(t-a) + \sin(t-a)\right) H_a(t) + 1 - \cos t + \sin t$$

This solution can only be guarenteed to be valid for $t \geq 0$, though as written it actually is valid for $t \geq -3$.

4. (20 points) Consider the ODE

$$\begin{cases} x''' - 27x = g(t) \\ x(0) = 1, x'(0) = 0, x''(0) = 0. \end{cases}$$

Determine a forcing $g(t)$ so that the solution $x(t)$ has the property that it grows larger, as $t \to \infty$, than any solution to the homogenous problem and larger than $g(t)$.

The characteristic polynomial is

$$P(s) = s^3 - 27 = (s-3)(s^2 + 3s + 9)$$

This means there are zeros at $s = 3$ and at $s = \frac{-3 \pm \sqrt{-27}}{2} = \frac{-3 \pm 3\sqrt{3}i}{2}$. Hence, the zero with the largest real part is $s = 3$.

If $X(s)$ is the Laplace transform of the solution then

$$P(s)X(s) + s^2 = G(s)$$

so

$$X(s) = \frac{G(s)}{P(s)} - \frac{s^2}{P(s)}$$

In particular, if $G(s)$ is rational and has a pole at $s = 3$ and no other pole with real part ≥ 3, then $X(s)$ will have at least a double pole at $s = 3$ and no other poles with real part ≥ 3. This means that $x(t)$ will grow more rapidly than $g(t)$ as $t \to \infty$. Similarly, since any solution to the homogenous problem has Laplace transform which has a simple pole at $s = 3$, $x(t)$ will grow faster than this solution.

A concrete example of such a $g(t)$ is $g(t) = e^{3t}$.

5. Consider the autonomous non-linear ODE

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} x_1 + x_2 \\ x_2 + x_1 \sin x_2 \end{pmatrix}.$$

(a) (15 points) For any $\alpha \in \mathbb{R}$ consider the initial conditions:

$$\begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} \alpha \\ 0 \end{pmatrix}.$$

Compute the the Picard iterates, $\mathbf{U}_k$, for these solutions and verify that they converge to a global solution to the IVP.

The first iterate is

$$\mathbf{U}_0(t) = \begin{pmatrix} \alpha \\ 0 \end{pmatrix}$$

Hence,

$$\mathbf{U}_1(t) = \begin{pmatrix} \alpha \\ 0 \end{pmatrix} + \int_0^t \begin{pmatrix} \alpha \\ 0 \end{pmatrix} ds = \begin{pmatrix} \alpha t \\ 0 \end{pmatrix}$$

We claim that

$$\mathbf{U}_k(t) = \begin{pmatrix} \alpha \sum_{j=0}^{k} \frac{t^j}{j!} \\ 0 \end{pmatrix}$$

Indeed, this is true for $\mathbf{U}_0$, and if it is true for $\mathbf{U}_k$, then

$$\mathbf{U}_{k+1}(t) = \begin{pmatrix} \alpha \\ 0 \end{pmatrix} + \int_0^t \begin{pmatrix} \alpha \sum_{j=0}^{k} \frac{s^j}{j!} \\ 0 \end{pmatrix} ds = \begin{pmatrix} \alpha \sum_{j=0}^{k+1} \frac{t^j}{j!} \\ 0 \end{pmatrix}$$

so the claim follows by induction. Clearly, as $k \to \infty$

$$\mathbf{U}_k \to \begin{pmatrix} \alpha e^t \\ 0 \end{pmatrix}$$

which are athe desired solutions.

(b) (5 points) For each α, determine the variational equation along the solutions found in part a)

The ODE is given by $\mathbf{X}' = \mathbf{F}(\mathbf{X})$ where

$$\mathbf{F}\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) = \begin{pmatrix} x_1 + x_2 \\ x_2 + x_1 \sin x2 \end{pmatrix}$$

The Jacobian is

$$D\mathbf{F} = \begin{pmatrix} 1 & 1 \\ \sin x_2 & 1 + x_1 \cos x_2 \end{pmatrix}.$$

Hence, the variational equations along the solutions

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} \alpha e^t \\ 0 \end{pmatrix}$$

found above are

$$\mathbf{Y}' = \begin{pmatrix} 1 & 1 \\ 0 & 1 + \alpha e^t \end{pmatrix} \mathbf{Y}.$$