Math 306, Fall 2014: Assignment #5

Due: **Wednesday, October 15th**

Instructions: Please ensure that your answers are legible and that sufficient steps are shown. Chapter numbers refer to the course text "Differential Equations, Dynamical Systems, and an Introduction to Chaos."

Problem #1. Chap. 6 # 1 parts g) and h). (Do others for practice if you need).

Problem #2. Chap. 6 # 3

Problem #3. Chap. 6 # 5

Problem #4. Chap. 6 # 7

Problem #5. Chap. 6 # 12 parts a), c), d), e) and h)

Problem #6. Chap. 6 # 13

Problem #7. Let A and B be $n \times n$ matrices.

a) Verify that flow the $\phi_t^A : \mathbb{R}^n \to \mathbb{R}^n$ of the linear system

$$\mathbf{Y}' = A\mathbf{Y}$$

is given by

$$\phi_t^A(\mathbf{x}) = e^{tA} \cdot \mathbf{x}.$$

b) Using the fact that $AB = BA$ implies $e^{A+B} = e^A e^B = e^B e^A$ show that if $AB = BA$, then

$$\phi_t^A \circ \phi_s^B = \phi_s^B \circ \phi_t^A.$$

c) Show that

$$\phi_t^A \circ \phi_s^A = \phi_{t+s}^A$$

Use this and the fact (which you may asssume) that for all t, $\mathbf{x} \mapsto \phi_t^A(\mathbf{x})$ is continuous, to conclude that this map is actually a homeomorphism.

Problem #8. Let A and B be $n \times n$ matrices.

a) Show that if $AB = BA$ and $\mathbf{v}$ is an eigenvector of A, then either $B\mathbf{v}$ is zero or $B\mathbf{v}$ is an eigenvector of A. Conversely, show that if $AB = BA$, B is invertible and $B\mathbf{v}$ an eigenvector of A, then $\mathbf{v}$ is an eigenvector of A.
b) Using a) show that if A has distinct real eigenvalues and $AB = BA$, then B has real eigenvalues and the same eigenvectors of A (though possibly different and not distinct eigenvalues).
c) Show that if A and B have non-zero entries only on the diagonal, then $AB = BA$.
d) Conclude that if A has distinct real eigenvalues, then $AB = BA$ if and only if there is a matrix T so that *both* $T^{-1}AT$ and $T^{-1}BT$ are in canonical form and this form is diagonal.

Bonus Problem. (Do not need to turn this in) Consider a pair of vectors $\mathbf{x}, \mathbf{v} \in \mathbb{R}^2$ with have $|\mathbf{v}| = 1$. We will think of these as a point $\mathbf{y}$ in $\mathbb{R}^4$ (i.e.

$$\mathbf{y} = (\mathbf{x}, \mathbf{v}) = \begin{pmatrix} x_1 \\ x_2 \\ v_1 \\ v_2 \end{pmatrix}$$

where

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \text{ and } \mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}.$$

Let X be the set of all such points. We will interpret X as the configuration space of a car (in an infinite parking lot), namely if $(\mathbf{x}, \mathbf{v}) \in X$, then $\mathbf{x}$ is the position of the front of the car and $\mathbf{v}$ is the direction the car is facing.

a) Let

$$D = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Show that $e^{tD}(\mathbf{x}, \mathbf{v}) = (\mathbf{x} + t\mathbf{v}, \mathbf{v})$ and so conclude $e^{tD} \in X$ if and only if $\mathbf{y} \in X$.

b) Let

$$T = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

show that $e^{tT}(\mathbf{x}, \mathbf{v}) = (\mathbf{x}, \mathbf{v}')$ where $\mathbf{v}'$ is obtained from $\mathbf{v}$ by rotating clockwise by t (counter-clockwise if t is negative) and so conclude $e^{tT} \in X$ if and only if $\mathbf{y} \in X$.

c) Check that $DT \neq TD$ and interpret this in terms of $e^{tT}e^{sD}(\mathbf{x}, \mathbf{v}) \neq e^{sD}e^{tT}(\mathbf{x}, \mathbf{v})$.

d) We model driving a car with *zero* turn radius as succesive applications of e^{tT} and e^{tD} (D is for drive and T is for turn). That is, one can drive from $(\mathbf{x}_0, \mathbf{v}_0)$ to $(\mathbf{x}_1, \mathbf{v}_1)$ if there are $t_1, \ldots, t_k$ and $s_1, \ldots, s_k$ (some possibly zero or negative) so that

$$(\mathbf{x}_1, \mathbf{v}_1) = e^{t_k T} e^{s_k D} \cdots e^{t_1 T} e^{s_1 D}(\mathbf{x}_0, \mathbf{v}_0).$$

Show, that for any two $(\mathbf{x}_0, \mathbf{v}_0)$ and $(\mathbf{x}_1, \mathbf{v}_1)$ there are t_1, s_1, t_2 so that

$$(\mathbf{x}_1, \mathbf{v}_1) = e^{t_2 T} e^{s_1 D} e^{t_1 T}(\mathbf{x}_0, \mathbf{v}_0).$$

In other words, its very easy to parallel park with a car with zero turn radius!

Bonus Problem. (Do not need to turn this in)

A better model of a car would be to consider

$$T_R = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

and

$$T_L = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

and driving from $(\mathbf{x}_0, \mathbf{v}_0)$ to $(\mathbf{x}_1, \mathbf{v}_1)$ to consist of $t_1, \ldots, t_k, r_1, \ldots, r_k$ and $s_1, \ldots, s_k$ so that

$$(\mathbf{x}_1, \mathbf{v}_1) = e^{t_k T_R} e^{r_k T_L} e^{s_k D} \cdots e^{t_1 T_R} e^{r_1 T_L} e^{s_1 D}(\mathbf{x}_0, \mathbf{v}_0).$$

a) Compute e^{tT_R} and e^{sT_L} and argue that they model the actual turning of a car (respectively, to the right and to the left). Verify that the turn radius is 1.

b) Show that for any $t \in [0, \pi/2)$ there is an r and s_1, s_2 so that

$$e^{tT_R} = e^{s_2 D} e^{r T_L} e^{s_1 D}$$

conclude that you can get to as many places driving with a car that can only turn to the left as driving a normal car.

c) Parallel park a car at $(0, \mathbf{E}_1)$ in a spot at $2\mathbf{E}_1$. That is, give a sequence of $t_1, \ldots, t_k, r_1, \ldots, r_k$ and $s_1, \ldots, s_k$ (allowed to be zero or negative) so that

$$(2\mathbf{E}_1, \mathbf{E}_1) = e^{t_k T_R} e^{r_k T_L} e^{s_k D} \cdots e^{t_1 T_R} e^{r_1 T_R L} e^{s_1 D}(0, \mathbf{E}_1).$$

(Hint: you should be able to do this in 3 maneuvers – i.e. $k = 3$).

d) For any integer $n \geq 1$ parallel park a car at $(0, \mathbf{E}_1)$ in a spot at $2n\mathbf{E}_1$. (Hint: you should be able to do this in $3n$ maneuvers).
e) Try and parallel park a car at $(0, \mathbf{E}_1)$ in a spot at $\mathbf{E}_1$ (this one is a bit trickier – try and do it in 4 maneuvers).
f) Parallel park a car it $(0, \mathbf{E}_1)$ in a spot at $x\mathbf{E}_1$ for arbitrary $x \in \mathbb{R}$.