Math 306, Fall 2014: Assignment #6

Due: **Wednesday, October 22nd**

Instructions: Please ensure that your answers are legible and that sufficient steps are shown. Chapter numbers refer to the course text "Differential Equations, Dynamical Systems, and an Introduction to Chaos."

Problem #1. Find a particular solution to the following second order ODEs:

a) $x'' + x = e^t$;
b) $x'' - x = e^t$.

Problem #2. Solve the IVP

$$\begin{cases} x'' + 4x = t \sin 2t \\ x(0) = 0, x'(0) = 1 \end{cases}$$

Problem #3. Consider mass on a spring whose motion is determined by

$$x'' + 5x' + 4x = 0.$$

a) Determine the initial conditions x_0, v_0 (i.e., so $x(0) = x_0$ and $x'(0) = v_0$) which allow one to stop the mass completely after time $t = \pi$ by a single blow with a hammer at time $t = \pi$ (i.e., with forcing $a\delta_\pi$.) Hint: Reverse time.
b) What about if you are allowed a second hammer blow at time $t = 2\pi$ and want to completely stop the mass after time $t = 2\pi$ (i.e., the forcing is $a\delta_\pi + b\delta_{2\pi}$)?

Problem #4. For $a > 0$, let $I_a(t) = H_0(t) - H_a(t)$, where $H_{t_0}(t)$ is the heaviside function which "turns on" at $t = t_0$.

a) Determine the $G_a(t)$ so that $I_a(t)$ is the solution to the IVP

$$\begin{cases} y' = G_a(t) \\ y(-1) = 0; \end{cases}$$

b) Determine the $G_a(t)$ so that $I_a(t)$ is the solution to the IVP

$$\begin{cases} y' = -y + G_a(t) \\ y(-1) = 0. \end{cases}$$

(Hint: You may use the fact that if $f(t)$ is a continuous function, then $f(t)\delta_{t_0}(t) = f(t_0)\delta_{t_0}(t)$).

Problem #5. For piecewise continuous $\mathbf{G}(t)$, consider the inhomogneous linear system

$$\mathbf{Y}' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mathbf{Y}' + \mathbf{G}(t).$$

a) Show that every non-zero solution to this equation is periodic with period 2π if $\mathbf{G}(t)$ is periodic with period 2π and

$$\int_0^{2\pi} \mathbf{G}(t) \cos t dt = \int_0^{2\pi} \mathbf{G}(t) \sin t dt = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

b) Show that if $\mathbf{G}(t)$ is periodic with period 2π and

$$\int_0^{2\pi} \mathbf{G}(t) \cos t dt = \mathbf{C} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ while } \int_0^{2\pi} \mathbf{G}(t) \sin t dt = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

then every solution satisfies

$$\mathbf{Y}(t) = \frac{t}{2\pi} \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \mathbf{C} + \mathbf{Y}_0$$

where $\mathbf{Y}_0$ is periodic with period 2π. How rapidly can $|\mathbf{Y}(t)|$ grow as $t \to \infty$ in this situation?

Problem #6. Compute

a) $\mathcal{L}\{\sin^2 t\}$;
b) $\mathcal{L}^{-1}\{\frac{3s+6}{s^2+9}\}$.

Problem #7.

a) Let f satisfy $f(t) = 0$ for all $t < 0$. Show that if $\mathcal{L}\{f(t)\} = F(s)$ is defined for $s > a$ and $\tau > 0$, then

$$\mathcal{L}\{f(t-\tau)\} = e^{-\tau s}F(s).$$

and this is defined for $s > a$.
b) What happens without the assumption on $f(t)$ for $t < 0$?
c) Suppose that $y(t)$ is s solution to the following "delayed" differential equation:

$$\begin{cases} y'(t) = y(t-1) + e^t \\ y(t) = 0, t \leq 0 \end{cases}$$

determine what the Laplace transform, $Y(s)$, of $y(t)$ is. You do not need to find y.

Problem #8. Let f be a non-negative piecewise continuous function and let $F(s)$ be its Laplace transform.

a) Show that if f is bounded by a constant C, then $F(s)$ is defined on at least $(0, \infty)$ and $F(s) \leq \frac{C}{s}$
b) Show that if there is a $C \geq 0$ so that $Ct^n \leq f(t)$ and $F(s)$ is defined on $(0, \infty)$, then, for such s,

$$\frac{Cn!}{s^{n+1}} \leq F(s).$$