Math 306, Fall 2014: Assignment #7

Due: **Wednesday, October 29th**

Instructions: Please ensure that your answers are legible and that sufficient steps are shown. Chapter numbers refer to the course text "Differential Equations, Dynamical Systems, and an Introduction to Chaos."

Problem #1. Suppose that $f(t)$ is piecewise continuous and and periodic with period T (i.e. so $f(t+T) = f(t)$ for all $t \in \mathbb{R}$). Show that

$$\mathcal{L}\{f(t)\} = \frac{1}{1-e^{-Ts}} \int_0^T e^{-st} f(t)dt$$

Problem #2. Use the Laplace transform to solve the following IVP

$$\begin{cases} x''' + x = g(t) \\ x(0) = 0, x'(0) = 1, x''(0) = 0 \end{cases}$$

where

$$g(t) = \begin{cases} 0 & t < \pi \\ \sin t & \pi < t < 3\pi/2 \\ -1 & t > 3\pi/2. \end{cases}$$

Problem #3. Recall the characteristic polynomial of the ODE

$$x^{(n)} + a_{n-1}x^{(n-1)} + \ldots + a_1 x' + a_0 = g(t)$$

is defined to be

$$P(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \ldots a_1\lambda + a_0.$$

Show that this is the same as the characteristic polynomial of the $n \times n$ matrix of the corresponding first order linear system.

Problem #4. Determine the asymptotic behavior of the following IVP as $t \to \infty$

$$\begin{cases} x'' - x = t \sin 2t + 2 \\ x(0) = 0, x'(0) = 1 \end{cases}$$

Problem #5. Determine the asymptotic behavior of the following IVP as $t \to \infty$

$$\begin{cases} x''' + x' = e^{-3t} + 2e^{-t} + t^2 \\ x(0) = 0, x'(0) = 0, x''(0) = 0 \end{cases}$$

Problem #6. Consider the IVP

$$\begin{cases} x''' - 8x = g(t) \\ x(0) = 1, x'(0) = 0, x''(0) = 0 \end{cases}$$

Determine a forcing $g(t)$ so that the resulting solution grows larger, as $t \to \infty$, than either $g(t)$ itself, or any solution to the homogenous problem.

Problem #7. Suppose that f, g and h are piecewise continuous functions with $|f(t)|, |g(t)|, |h(t)| \le Ce^{\alpha t}$ for some $\alpha \in \mathbb{R}$. Define the *convolution* of f and g to be the functions

$$(f \star g)(t) = \int_0^t f(\tau)g(t-\tau)d\tau$$

a) Show that $f \star g = g \star f$;

b) Show that $(f \star g) \star h = f \star (g \star h)$;
c) Show that

$$\mathcal{L}\{f \star g\} = \mathcal{L}\{f\} \cdot \mathcal{L}\{g\}.$$

i.e., the Laplace transform of a convolution of f and g is the product of the Laplace transform of f and of g separately.
d) Use this to compute

$$\mathcal{L}^{-1}\left\{\frac{1}{(s^2 + a^2)^2}\right\}.$$

Problem #8. Use the Laplace transform to prove the variation of parameters formula for a solution to the ODE $y' = ay + g(t)$. (Hint: Use the previous exercise).