Math 306, Fall 2014: Assignment #8

Due: **Wednesday, November 12th**

Instructions: Please ensure that your answers are legible and that sufficient steps are shown. Chapter numbers refer to the course text "Differential Equations, Dynamical Systems, and an Introduction to Chaos."

Problem #1. Chapter 7, # 2

Problem #2. Chapter 7, # 3

Problem #3. Chapter 7, # 7

Problem #4. Chapter 8, # 1 (ii), (iii), (iv)

Problem #5. Consider the autonomous ODE

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}' = \begin{pmatrix} x_1 + x_2 - (x_1^3 + x_2^2 x_1) \\ -x_1 + x_2 - (x_1^2 x_2 + x_2^3) \\ 2x_3(1 - x_3) \end{pmatrix}.$$

a) Use cylinderical coordinates to find all solutions of this ODE.
b) Determine each solution, $\mathbf{X}(t)$, which has $|\mathbf{X}(t)|$ constant and compute the variational equation along each of these equations.

Problem #6. Let $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$ be C^1. Suppose that the autonomous system $\mathbf{X}' = \mathbf{F}(\mathbf{X})$ admits a global solution $\mathbf{X}(t)$ with $\mathbf{X}(t) = C \cos t\mathbf{E}_1 + C \sin t\mathbf{E}_2$ for some $C > 0$, i.e., $\mathbf{X}(t)$ parameterizes a circle.

a) Show that if $n = 2$, and $|\mathbf{X}_0| < C$, then the solution to the IVP

$$\begin{cases} \mathbf{X}' = \mathbf{F}(\mathbf{X}) \\ \mathbf{X}(0) = \mathbf{X}_0 \end{cases}$$

must satisfy $|\mathbf{X}(t)| < C$ (and so by the theorem on pg. 146 is a global solution.) (Hint: Use local uniqueness and the intermediate value theorem).
b) Given an example to show that this is no longer true for $n \geq 3$.

Problem #7. Show that if $u : [a, b] \to \mathbb{R}$ is a C^1 function that satisfies the differential *inequality*

$$u' \leq \mu u + g(t),$$

where g is continuous, then, for $t \in [a, b]$,

$$u(t) \leq u(a)e^{\mu(t-a)} + \int_a^t e^{\mu(t-s)} g(s) ds.$$

Be careful about how signs affect inequalities!

Problem #8. Show that if $u : [a, b) \to \mathbb{R}$ is a *positive* C^1 function that satisfies the differential *inequality*

$$u' \geq \mu u^2,$$

for $\mu > 0$, then we must have $b \leq a + \frac{1}{u(a)\mu}$. (Hint: Show that $u(t) \geq \frac{u(a)}{1-\mu u(a)(t-a)}$)).

Problem #9. Use Problem # 7, the Cauchy-Schwarz inequality

$$|\mathbf{X} \cdot \mathbf{Y}| \leq |\mathbf{X}||\mathbf{Y}|$$

and the theorem on pg. 146 to show that if

$$\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$$

is C^1 and satisfies $|\mathbf{F}(X)| \leq C|\mathbf{X}|$ for some $C > 0$, then the IVP

$$\begin{cases} \mathbf{X}' = \mathbf{F}(\mathbf{X}) \\ \mathbf{X}(0) = \mathbf{X}_0 \end{cases}$$

has a global solution. (Hint: Consider $u(t) = |\mathbf{X}(t)|^2$).