Homework 2 Sample Solutions

Problem 2.3. In Figure 2.2 (not pictured here), you see four direction fields. Match each of these direction fields with one of the systems in the previous exercise.

Solution. Many of you just matched the pictures with the systems, with no explanation. That will not earn you full credit in this class. One needs to explain what one is thinking when answering a homework question.

To answer this question, one simply needs to identify the eigenvalues and associated eigenvectors of each system (which you all did in problem 2) and then identify which direction field represents these eigenvectors and eigenvalues.

In my opinion, this obvious direction field to start with is 2. It has a line with no arrows on it, which clearly represents eigenvectors with eigenvalue 0. The only system we are considering which had an eigenvalue of 0 is (b), so 2 matches (b).

The next obvious direction field to consider is 4. It's plain that all eigenvalues associated with this direction field are positive. The only such system from problem 2 is (a). Moreover, the eigenvectors of (a) were $(1, 0)$ and $(1, 1)$, both of which are clealy eigenvectors in this direction field.

Looking at the remaining direction fields, 1 has an obvious negative eigenvalue with eigenvector $(1, -1)$. This matches (c). Likewise 3 has a positive eigenvalue with associated eigenvector $(2, 1)$ or so. System (d) has a positive eigenvalue with eigenvector $(2, \sqrt{10} - 2) \approx (2, 1)$, so this matches. □

Problem 2.8. Describe all possible 2×2 matrices with eigenvalues of 0 and 1.

Solution. A number of you either weren't sure how to attack this problem or weren't careful with the details as you found your solution. I'll just dive in to my solution.

We start with a matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Since one of the eigenvalues is 0, we know that the determinant of A is zero, i.e. $ad - bc = 0$. We also know that since 1 is an eigenvalue, $\det(A - I) = 0$ too. Writing this out, we find

$$(a - 1)(d - 1) - bc = ad - a - d + 1 - bc = 0$$

We can substitute $ad - bc = 0$ to obtain

$$-a - d + 1 = 0 \implies d = 1 - a$$

Using this, we now have

$$(a - 1)(d - 1) - bc = (a - 1)(-a) - bc = 0 \implies bc = a - a^2$$

At this point some of you were tempted to write

$$c = \frac{a - a^2}{b}$$

but this is only true *if* $b \neq 0$*!!* Hence, we need to split this problem into two cases: Either $b \neq 0$ or $b = 0$. In the former case, we are done. We have shown that such matrices are of the form

$$\left\{ \begin{pmatrix} a & b \\ \frac{a-a^2}{b} & 1-a \end{pmatrix} : a \in \mathbb{R}, b \in \mathbb{R} \setminus \{0\} \right\}$$

and it is easy to check that all matrices of this form *do indeed* have eigenvalues $0, 1$.

It remains to determine what happens if $b = 0$. In this case, we get

$$(0)c = 0 = a - a^2 \implies a = 0 \text{ or } 1$$

Hence the remaining matrices are of the form

$$\left\{ \begin{pmatrix} a & 0 \\ c & 1-a \end{pmatrix} : a = 0 \text{ or } 1, c \in \mathbb{R} \right\}$$

Again, it's easy to check that all matrices of this form do have eigenvalues $0, 1$. This completes the problem. □

Problem 2.11. Prove that two vectors $V = (v_1, v_2)$ and $W = (w_1, w_2)$ are linearly independent if and only if

$$\det \begin{pmatrix} v_1 & w_1 \\ v_2 & w_2 \end{pmatrix} \neq 0.$$

Solution. This was another problem that students tended to stumble over by dividing by unknown constants, hence assuming that they are nonzero, etc. Whenever we divide by something, we are implicitly assuming that it is nonzero. Hence, we must separately address the case in which it is zero. You'll see me do just that in what follows:

Here I prove the contrapositive of the statement. That is, I prove that the vectors are linearly dependent iff the determinant is zero. So, to begin, suppose that V, W are linearly dependent. This means that there exist numbers $c_1, c_2 \in \mathbb{R}$ such that $c_1 V + c_2 W = 0$, but at least one of c_1, c_2 is nonzero (but one of them could be zero!). Suppose that $c_1 \neq 0$. Then we have $V = -\frac{c_2}{c_1} W$, and so

$$\det \begin{pmatrix} v_1 & w_1 \\ v_2 & w_2 \end{pmatrix} = \det \begin{pmatrix} -c_2 w_1/c_1 & w_1 \\ -c_2 w_2/c_1 & w_2 \end{pmatrix} = -\frac{c_2 w_1 w_2}{c_1} + \frac{w_1 c_2 w_2}{c_1} = 0$$

If, on the other hand, $c_1 = 0$, then by assumption $c_2 \neq 0$ (since both cannot be zero), and a similar calculation shows the determinant is zero.

Conversely, suppose that

$$\det \begin{pmatrix} v_1 & w_1 \\ v_2 & w_2 \end{pmatrix} v_1 w_2 - v_2 w_1 = 0$$

Then $v_1 w_2 = v_2 w_1$. Thus, we have that

$$w_2 \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} v_1 w_2 \\ v_2 w_2 \end{pmatrix} = \begin{pmatrix} v_2 w_1 \\ v_2 w_2 \end{pmatrix} = v_2 \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

Thus, we've shown that

$$w_2 V - v_2 W = 0$$

If v_2 or w_2 is nonzero, then by definition V, W are linearly dependent. Otherwise we have $v_2 = w_2 = 0$, and by a similar process to the above, we can show

$$w_1 V - v_1 W = 0$$

Thus, again, we're done if either v_1 or w_1 is nonzero. If this is not the case, then we have that $v_1 = v_2 = w_1 = w_2 = 0$, which implies that both V and W are the zero vector, hence we trivially have $V + W = 0$, implying that they are linearly dependent. This completes the proof.

As you can see, a competely rigorous and correct proof of this fact is more involved than you might have thought at first! Be sure to pay attention to the small details in your future assignments. □

Problem 2.14. Prove that the eigenvectors of a 2×2 matrix corresponding to distinct real eigenvalues are always linearly independent.

Solution. This is another problem in which students had trouble finding the right attack strategy. The correct solution is actually quite simple.

Let v_1, v_2 be nonzero eigenvectors corresponding to the distinct real eigenvalues λ_1, λ_2 of the matrix A. Suppose that $c_1 v_1 + c_2 v_2 = 0$. Applying A to this equation, we get

$$c_1 A v_1 + c_2 A v_2 = c_1 \lambda_1 v_1 + c_2 \lambda_2 v_2 = 0$$

On the other hand, multiplying by λ_1 yields

$$\lambda_1 (c_1 v_1 + c_2 v_2) = c_1 \lambda_1 v_1 + c_2 \lambda_1 v_2 = 0$$

Subtracting these equations yields

$$c_2 (\lambda_2 - \lambda_1) v_2 = 0$$

But since $v_2 \neq 0$ and the eigenvalues are distinct (i.e. $\lambda_2 - \lambda_1 \neq 0$), this implies that $c_2 = 0$. We similarly obtain $c_1 = 0$. Thus, by definition, v_1, v_2 are linearly independent.

(If you're not sure why this says that v_1, v_2 are linearly independent, you should look up the definition in a linear algebra book.) □