Homework 3 Sample Solutions

Problem 3.1. In Figure 3.9 (not pictured here), you see six phase portraits. Match each of these phase portraits with one of the following linear systems (listed below).

Solution. Notice that each of the given diagrams has noticeably different geometry from each of the others. Two are spiral sinks, two are spiral sources, and two are centers. In each of these groups of two, one is going clockwise while the other is going counterclockwise. This is all the information we need to gather from the given systems to match them up.

(a) $A = \begin{pmatrix} 3 & 5 \\ -2 & -2 \end{pmatrix}$. In this case, the eigenvalues are $\frac{1 \pm \sqrt{15}i}{2}$. Thus the eigenvalues have positive real part, so the solutions will be spiral sources. Now, to check whether it's clockwise or counterclockwise, simply plug in the vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. We obtain $\begin{pmatrix} 3 & 5 \\ -2 & -2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$. This has a negative y component, so the tangent vector to our solution is pointing down when it crosses the x-axis. Hence, solutions are going clockwise. This matches diagram 3.

(b) $A = \begin{pmatrix} -3 & -2 \\ 5 & 2 \end{pmatrix}$. We get eigenvalues $\frac{-1 \pm \sqrt{15}i}{2}$, hence our solutions are spiral sinks. Plugging in $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ to A, we see that solutions go counterclockwise.

The rest are similar. $\square$

Problem 3.14. Suppose the 2×2 matrix A has repeated eigenvalues λ. Let $V \in \mathbb{R}^2$. Using the previous problem, show that either V is an eigenvector for A or else $(A - \lambda I)V$ is an eigenvector for A.

Solution. I think some students had trouble with this one because they tried to construct explicit formulas for eigenvalues of A, or something similarly formulaic and tedious. For a problem like this, one should try to use the *algebraic* properties that we know of. In particular, as the problem suggests, we should use that A satisfies its characteristic equation.

For a repeated eigenvalue in the 2×2 case, we have that the characteristic equation for A is $(x - \lambda)^2 = 0$. Thus, we have that $(A - \lambda I)^2 = 0$ as matrices. Thus for any $V \in \mathbb{R}^2$, $(A - \lambda I)^2 V = 0$. We can split this up a bit:

$$(A - \lambda I)(A - \lambda I)V = 0$$

which implies

$$A(A - \lambda I)V = \lambda I(A - \lambda I)V = \lambda(A - \lambda I)V$$

But by definition, this says that $(A - \lambda I)V$ is an eigenvector of A! The only subtlety here is that $(A - \lambda I)V$ might be 0. In this case, either $V = 0$ or else V is a nonzero eigenvector of A.

Note that we have shown this very generally. We let V be *any* vector in $\mathbb{R}^2$. Your proof should similarly not make any assumptions about what V is. $\square$