Homework 4 Sample Solutions

Problem 4.6. Prove that any two linear systems with the same eigenvalues $\pm i\beta$,$\beta \neq 0$ are conjugate. What happens if the systems have eigenvalues $\pm i\beta$ and $\pm i\gamma$ with $\beta \neq \gamma$? What if $\gamma = -\beta$?

Solution. People actually generally got this one right, but I thought I'd do it anyway because it illustrates both how to find conjugates and how to prove conjugates do not exist.

I'll denote the systems as $X' = AX$ and $Y' = BY$ throughout this solution. Firstly, if the two systems both have eigenvalues $\pm i\beta$ with $\beta \neq 0$, then by the work in the book, there exist invertible matrices T and S such that

$$T^{-1}AT = S^{-1}BS = \begin{pmatrix} 0 & -\beta \\ \beta & 0 \end{pmatrix} = C$$

which corresponds to a new system, $Z' = CZ$. Then the map T^{-1} sends solutions of $X' = AX$ to solutions of $Z' = CZ$, and the map S sends solutions of $Z' = CZ$ to solutions of $Y' = BY$. I therefore claim that the composition $S \circ T^{-1}$ is the desired conjugacy. Since it sends solutions of the first system to solutions of the second, it suffices to show that $S \circ T^{-1}$ is a homeomorphism of $\mathbb{R}^2$. But since it's clearly an invertible linear map, it's a homeomorphism by what we talked about in section. This completes the proof.

(By the way, if you didn't explain that $S \circ T^{-1}$ is a homeomorphism on your homework, that's OK. I just wanted to give all the details here.)

Now if A has eigenvalues $\pm i\beta$ and B has eigenvalues $\pm i\gamma$, and $\gamma = \pm\beta$, then we are in the above situation, hence the systems are conjugate.

It remains to see what happens if $|\beta| \neq |\gamma|$. (I assume here that $\gamma \neq 0$ too.) Suppose without loss of generality that $|\beta| > |\gamma|$. Then we have previously found that solutions of the first system look like linear combinations of $\sin\beta t$ and $\cos\beta t$, while solutions of the second look like linear combinations of $\sin\gamma t$ and $\cos\gamma t$. For this reason, nonconstant solutions of the first system have period $\frac{2\pi}{|\beta|}$ while nonconstant solutions of the second have period $\frac{2\pi}{|\gamma|}$.

Now suppose that $h(x, y)$ is a conjugacy from the first system to the second. By definition, this means that $\phi^B(t, h(X_0)) = h(\phi^A(t, X_0))$ for all $X_0 \in \mathbb{R}^2$. But notice that we always have

$$\phi^A(t, X_0) = \phi^A(t + \frac{2\pi}{|\beta|}, X_0)$$

for all t and any X_0. Since h is a *function* (let alone a homeomorphism), this would force

$$\phi^B(t, h(X_0)) = \phi^B(t + \frac{2\pi}{|\beta|}, h(X_0))$$

But this is impossible since solutions of the second system have a period of $\frac{2\pi}{|\gamma|}$, which is longer than $\frac{2\pi}{|\beta|}$! Hence no such conjugacy can exist.

This proves that if $|\beta| \neq |\gamma|$, then the systems are not conjugate. □

Problem 5.5. Put the following matrices in canonical form: (listed below).

Solution.

(a) $A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

When a problem asks you to put some matrix A in another form, what they mean is for you to find an invertible T such that $T^{-1}AT$ is of the desired form. If you do not write down what this T is, you have not completed the problem.

To start, we find eigenvalues and eigenspaces. For this matrix, we get eigenvalues ± 1 and eigenspaces $E_1 = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$ and $E_{-1} = \text{span} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ (your bases may look different). Since we can find a basis for $\mathbb{R}^3$ consisting of eigenvectors, A can be diagonalized. Choosing $T = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix}$, we get $T^{-1}AT = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$.

(b) $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

This time the only eigenvalue is 1, with eigenspace $E_1 = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$. This time we cannot find a basis of eigenvectors of the matrix. Rather, we solve

$$(A - I)X = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

(Similarly, we could have chosen any eigenvector for the right hand side.) This is a straightforward process, which yields infinitely many solutions. One solution is $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. This, with two vectors from E_1, completes the basis of $\mathbb{R}^3$, yielding the matrix $T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ (notice the order of the basis we've used here). Thus, we have $T^{-1}AT = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

(c) $A = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}$

This time we get eigenvalues $1, \pm i$ with eigenspaces $E_1 = \text{span}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ and $E_i = \text{span}\begin{pmatrix} i \\ -1 \\ 1 \end{pmatrix}$ (there's no need to write down E_{-i}). Using real and imaginary parts of the eigenvector for i, we get a basis for $\mathbb{R}^3$ which yields the matrix $T = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 1 & 0 \end{pmatrix}$. Thus, we obtain $T^{-1}AT = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$. □

Problem 5.11. Show that if A and/or B are noninvertible matrices, then AB is also noninvertible.

Solution. This is the question which the most people got wrong. A typical incorrect answer goes as follows: We know A is noninvertible iff $\det A = 0$, and we also know $\det(AB) = (\det A)(\det B)$. Thus if $\det A = 0$ or $\det B = 0$, then AB is not invertible.

But this is circular reasoning! Why? Because we haven't proved that $\det(AB) = (\det A)(\det B)$ yet! In fact, if you check on page 81 of the text where the authors prove this fact, they delay the case of A or B being noninvertible to this very exercise! They only give the proof for the case in which both matrices are invertible. This exercise completes the proof, so obviously we cannot use that fact to prove itself!

Rather, we must take a more hands-on approach. It follows from the rank-nullity theorem that a square matrix A is invertible iff it is one-to-one iff it is onto (when viewed as a linear map $\mathbb{R}^n \to \mathbb{R}^n$). Thus, if we assume that A is not invertible, it follows that A is not onto. But then AB cannot be onto either (you should check this yourself). Thus AB is not invertible.

Similarly, if we assume that B is not invertible, then it follows that B is not one-to-one (again by the rank-nullity theorem). But then AB cannot be one-to-one either (again, check this yourself), whence it is not invertible. This completes the proof.

You could also use the property that an invertible matrix A has a unique solution to the equation $AX = Y$ for every $Y \in \mathbb{R}^n$. The proof would look similar in structure to the one given here. □