# Homework 5 Sample Solutions

**Problem 6.12.** Compute the exponentials of the following matrices. (Listed below)

*Solution.* I'm not going to do all of these, but I'll pick a representative sample.

(a) $A = \begin{pmatrix} 5 & -6 \\ 3 & -4 \end{pmatrix}$

With matrices like this one, it's easiest to first make a change of basis to put it in canonical form. The exponentials of $2 \times 2$ matrices in canonical form is all done in the book, so this will make our life easy.

The usual computation reveals that this matrix has eigenvalues $2, -1$ with eigenspaces $E_2 = \text{span}\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $E_{-1} = \text{span}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Choosing $T = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$, we have that

$B = T^{-1}AT = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}$. As computed in the book, we have that $e^B = \begin{pmatrix} e^2 & 0 \\ 0 & e^{-1} \end{pmatrix}$.

Moreover, by the proposition on page 126, $e^A = Te^B T^{-1} = \begin{pmatrix} 2e^2 - e^{-1} & 2e^{-1} - 2e^2 \\ e^2 - e^{-1} & 2e^{-1} - e^2 \end{pmatrix}$.

(e) $A = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{pmatrix}$

This is actually quite an easy matrix to exponentiate since it is what we call a *nilpotent* matrix. Nilpotent just means that some power of this matrix is 0, thus the infinite sum defining the exponential becomes finite! We compute: $A^2 = \begin{pmatrix} 0 & 0 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $A^3 = 0$. Thus,

by definition, $e^A = I + A + A^2/2 + 0 + 0 + \ldots = \begin{pmatrix} 1 & 1 & 7/2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}$. Take advantage of this

trick whenever you can!

(h) $A = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$

Since this matrix does not have real coefficients, we cannot use our usual tricks to put it in canonical form (so trying to take real and imaginary components of eigenvectors to get $T$ will not work). However, notice that it is already in its canonical form! It is diagonalized after all, and diagonal matrices are as nice as they come. In fact, exponentiating this matrix works exactly the same as exponentiating a diagonal matrix with real components. The proof of the formula is exactly the same as the proof given in the book.

Thus, we have that $e^A = \begin{pmatrix} e^i & 0 \\ 0 & e^{-i} \end{pmatrix} = \begin{pmatrix} \cos 1 + i \sin 1 & 0 \\ 0 & \cos 1 - i \sin 1 \end{pmatrix}$. $\square$

**Problem 6.13.** Find an example of two matrices $A, B$ such that

$$e^{A+B} \neq e^A e^B$$

*Solution.* Recall that if $AB = BA$, then $\exp(A + B) = \exp(A)\exp(B) = \exp(B)\exp(A)$. Thus if we want to find an example of the above, we had better make sure that $A$ and $B$ don't commute!

Here's an example: $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. You can check for yourself that $AB \neq BA$. So let's try and see if this satisfies the required property.

Via the same method as Problem 6.12, I calculate

$$e^A = \begin{pmatrix} e & e \\ 0 & e \end{pmatrix}, e^B = \begin{pmatrix} e & 0 \\ e & e \end{pmatrix}, e^{A+B} = \frac{1}{2}\begin{pmatrix} e + e^3 & e^3 - e \\ e^3 - e & e + e^3 \end{pmatrix}$$

On the other hand, $e^A e^B = \begin{pmatrix} 2e^2 & e^2 \\ e^2 & e^2 \end{pmatrix}$. Thus these matrices have the desired property.

Honestly, these were the first two matrices I could think of which did not commute. It turned out that they satisfied the problem as well, but I didn't know that in advance. $\square$

**Problem 8.** (Corrected statement)

Let $A, B$ be $n \times n$ matrices.

a) Show that if $AB = BA$ and $\mathbf{v}$ is an eigenvector of $A$, then either $B\mathbf{v}$ is zero or $B\mathbf{v}$ is an eigenvector of $A$. Conversely, show that if $AB = BA$, $B$ is invertible and $B\mathbf{v}$ is an eigenvector of $A$, then $\mathbf{v}$ is an eigenvector of $A$.

b) Using a) show that if $A$ has distinct real eigenvalues and $AB = BA$, then $B$ has real eigenvalues and the same eigenvectors of $A$.

c) Show that if $A$ and $B$ have non-zero entries only on the diagonal, then $AB = BA$.

d) Conclude that if $A$ has distinct real eigenvalues, then $AB = BA$ if and only if there is a matrix $T$ so that *both* $T^{-1}AT$ and $T^{-1}BT$ are in canonical form, and this form is diagonal.

*Solution.*

a) First, assume that $A\mathbf{v} = \lambda\mathbf{v}$ for some $\lambda$ (i.e. $\mathbf{v}$ is an eigenvector of $A$). Then applying $B$ to both sides, we obtain $BA\mathbf{v} = B(\lambda\mathbf{v}) \implies AB\mathbf{v} = \lambda B\mathbf{v}$, which is to say that $B\mathbf{v}$ is an eigenvector of $A$.

Conversely, suppose that $B$ is invertible and $B\mathbf{v}$ is an eigenvector of $A$, i.e. $AB\mathbf{v} = \lambda B\mathbf{v}$ for some $\lambda$. Then $BA\mathbf{v} = AB\mathbf{v} = B(\lambda\mathbf{v})$. Applying $B^{-1}$ to both sides, we obtain $B^{-1}BA\mathbf{v} = A\mathbf{v} = B^{-1}B\lambda\mathbf{v} = \lambda\mathbf{v}$, i.e. $\mathbf{v}$ is an eigenvector of $A$.

b) Since $A$ has distinct real eigenvalues, each of its eigenspaces is one dimensional. More-over, whenever $\mathbf{v}$ is a (nonzero) eigenvector of $A$, part a) implies that $B\mathbf{v}$ is a (nonzero) eigenvector of $A$ as well, with the same eigenvalue. Thus $B\mathbf{v}$ and $\mathbf{v}$ live in the same one dimensional vector space, i.e. $B\mathbf{v} = \lambda'\mathbf{v}$ for some real $\lambda'$. Thus $B$ has the same eigenvec-tors as $A$, and all of its eigenvalues are real. (Be careful on this point: $A$ has $n$ distinct

eigenvalues, and each gave rise to a different eigenvector of $B$. But $B$ can have at most $n$ linearly independent eigenvectors, so the eigenvalues obtained in this way must be *all* of $B$'s eigenvalues. We therefore saw that they were all real.)

c) This is very easy to see. Just write down two generic diagonal matrices and you will see that they must commute.

d) First suppose that $AB = BA$. Then by part b) $A$ and $B$ have the same eigenvectors. Moreover since $A$ has distinct real eigenvalues, it has $n$ linearly independent eigenvectors. Call them $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$. Let $T$ be the matrix whose $i^{\text{th}}$ column is $\mathbf{v}_i$ for all $i$ (that is, $TE_i = \mathbf{v}_i$). Since the $\mathbf{v}_i$ are also the eigenvectors of $B$, it follows that $T^{-1}AT$ and $T^{-1}BT$ are both diagonal matrices (with their eigenvalues on the diagonal). In this situation, we say that $A$ and $B$ are *simutaneously diagonalized*, by the way.

Conversely, suppose that there is a $T$ such that $T^{-1}AT$ and $T^{-1}BT$ are diagonal. Then we easily have the following:

$$T^{-1}ABT = (T^{-1}AT)(T^{-1}BT) = (T^{-1}BT)(T^{-1}AT) = T^{-1}BAT$$

All I have used here is that I can add a $TT^{-1} = I$ wherever I want, and that diagonal matrices commute (see c) above). Multiplying each side on the left by $T$ and on the right by $T^{-1}$ yields $AB = BA$ as claimed. $\qquad\square$