Homework 6 Sample Solutions

Problem #1. Find a particular solution of the following second order ODEs:
a) $x'' + x = e^t$
b) $x'' - x = e^t$

Solution. Several people used a complicated variation of parameters method to solve these, so I thought I'd illustrate how much easier it is to use undetermined coefficients instead.

a) Note that in this case our characteristic polynomial (obtained from setting $x = e^{rt}$ and plugging in to the homogeneous equation) is $r^2 + 1 = 0$. Thus the system has eigenvalues $\pm i$. Since 1 is not an eigenvalue, we simply try the solution Ae^t (since the RHS is just e^t). Plugging this in, we obtain $Ae^t + Ae^t = e^t$. Thus $A = 1/2$, and we have the particular solution $\frac{1}{2}e^t$ (See how easy that was? No integrals required!).

b) Now our characteristic polynomial has become $r^2 - 1 = 0$, so we have eigenvalues ± 1. Since 1 is an eigenvalue with multiplicity 1, we modify our guess. That is, try Ate^t. Plugging in yields $2Ae^t + Ate^t - Ate^t = e^t$. So once again $A = 1/2$ works, and we have the particular solution $\frac{1}{2}te^t$. □

Problem #3. Consider a mass on a spring whose motion is determined by

$$x'' + 5x' + 4x = 0.$$

a) Determine the initial conditions x_0, v_0 which allow one to stop the mass completely after time $t = \pi$ by a single blow with a hammer at time $t = \pi$ (i.e., with forcing $a\delta_\pi$.)

b) What about if you are allowed a second hammer blow at time $t = 2\pi$ and want to completely stop the mass after time $t = 2\pi$ (i.e., the forcing is $a\delta_\pi + b\delta_{2\pi}$)?

Solution.

a) A lot of people made mistakes on this one, by either making the wrong change of variables or getting an incorrect differential equation from their change of variables. Here's what I did:

The condition that the mass stops completely after time $t = \pi$ is equivalent to setting the initial condition $x(\pi + 1) = x'(\pi + 1) = 0$ (or at any time after $t = \pi$). I don't want to think too hard about what will actually happen at $t = \pi$, so I'll use this instead.

Now, let's make a change of variables to make our lives easier. I set $y(t) = x(\pi + 1 - t)$. That way $y(0) = y'(0) = 0$. Also, $y'(t) = -x'(\pi + 1 - t)$, but $y''(t) = x''(\pi + 1 - t)$. Thus, y satisfies the differential equation

$$y'' - 5y' + y = a\delta_\pi(\pi + 1 - t) = a\delta_0(1 - t) = a\delta_0(t - 1) = a\delta_1(t)$$

Laplacing everything, we end up with the expression

$$Y(s) = \mathcal{L}\{Y\} = \frac{ae^{-s}}{s^2 - 5s + 4} = \frac{ae^{-s}}{3}\left(\frac{1}{s-4} - \frac{1}{s-1}\right)$$

using partial fractions. Inverse Laplacing yields $y(t) = \frac{a}{3}H_1(t)\left(e^{4(t-1)} - e^{t-1}\right)$. Thus $x(t) = \frac{a}{3}(1 - H_\pi(t))(e^{4(\pi-t)} - e^{\pi-t})$. Differentiating at every $t \neq \pi$, we have $x'(t) = \frac{a}{3}(1 - H_\pi(t))(e^{\pi-t} - 4e^{4(\pi-t)})$. Thus $x(0) = x_0 = \frac{a}{3}(e^{4\pi} - e^{\pi})$ and $x'(0) = v_0 = \frac{a}{3}(e^{\pi} - 4e^{4\pi})$.

b) We take the same strategy, but instead use the initial condition $x(2\pi+1) = x'(2\pi+1) = 0$ and the change of variables $y(t) = x(2\pi + 1 - t)$. Then our IVP is $y(0) = y'(0) = 0$ and

$$y' - 5y' + 4y = a\delta_{\pi+1} + b\delta_1$$

The same process yields

$$x(t) = \frac{1}{3}\left(a\left(1 - H_\pi(t)\right)\left(e^{4(\pi-t)} - e^{\pi-t}\right) + b\left(1 - H_{2\pi}(t)\right)\left(e^{4(2\pi-t)} - e^{2\pi-t}\right)\right)$$

Thus we have $x(0) = x_0 = \frac{a}{3}(e^{4\pi} - e^{\pi}) + \frac{b}{3}(e^{8\pi} - e^{2\pi})$ and $x'(0) = v_0 = \frac{a}{3}(e^{\pi} - 4e^{4\pi}) + \frac{b}{3}(e^{2\pi} - 4e^{8\pi})$. □

Problem #8. Let f be a non-negative piecewise continuous function and let $F(s)$ be its Laplace transform.

a) Show that if f is bounded by a constant C, then $F(s)$ is defined on at least $(0, \infty)$ and $F(s) \leq \frac{C}{s}$.

b) Show that if there is a $C \geq 0$ so that $Ct^n \leq f(t)$ and $F(s)$ is defined on $(0, \infty)$, then, for such s,

$$\frac{Cn!}{s^{n+1}} \leq F(s)$$

Solution.

a) I was kind of surprised to see so many people not even use the fact that f is non-negative in their solutions to this problem. It is absolutely essential to use this; otherwise the statement is not true! I'll show you what I mean.

By definition, $F(s) = \int_0^\infty e^{-st}f(t)dt$. On the one hand, $f \leq C$ implies that $\int_0^\infty e^{-st}f(t)dt \leq \int_0^\infty e^{-st}Cdt = \mathcal{L}\{C\} = \frac{C}{s}$ for any $s > 0$. On the other hand, $f \geq 0$ implies that $0 = \int_0^\infty 0dt \leq \int_0^\infty e^{-st}f(t)dt$. Thus, we have that $0 \leq F(s) \leq \frac{C}{s}$ for every $s > 0$. It is only because we have both inequalities that we can conclude that $F(s)$ converges for all $s > 0$.

b) This just comes down to doing integration by parts a whole bunch. We have that

$$F(s) = \int_0^\infty e^{-st}f(t)dt \geq \int_0^\infty Ce^{-st}t^n dt = -\frac{Ct^n e^{-st}}{s}\Bigg|_0^\infty + \frac{n}{s}\int_0^\infty Ct^{n-1}e^{-st}dt$$

The left term is 0 since we are assuming $s > 0$ and we proceed with the same method with the right term. Eventually we get

$$F(s) \geq \frac{n!}{s^n} \int_0^\infty Ce^{-st} dt = \frac{Cn!}{s^{n+1}}$$

which is exactly what we wanted. $\square$