Homework 7 Sample Solutions

Problem #5. Deterime the asymptotic behavior of the following IVP as $t \to \infty$

$$\begin{cases} x''' + x' = e^{-st} + 2e^{-t} + t^2 \\ x(0) = 0, x'(0) = 0, x''(0) = 0 \end{cases}$$

Solution. This solution also serves as a bit of an apology; I didn't cover this technique in section, and I think I may have misled you to think that, in this situation, we cannot actually determine the asymptotic behavior of a problem like this. That being said, there's no excuse for not reading the handout, so I can't say a problem like this is unfair. I just may have accidentally put you at a disadvantage here. Like I said, I apologize. Anyway, let's get on to the actual solution.

As usual in this type of situation, we take the Laplace transform of both sides. This yeilds the following.

$$(s^3 + s)\mathcal{L}\{x\} = \frac{1}{s+3} + \frac{2}{s+1} + \frac{2}{s^3}$$

for any $s > 0$. Hence

$$\mathcal{L}\{x\} = \left(\frac{1}{s+3} + \frac{2}{s+1} + \frac{2}{s^3}\right) \Big/ (s(s^2+1))$$

One can read off from this expression that $\mathcal{L}\{x\}$ has poles at $-3, -1, \pm i$, and 0. Furthermore, the limit as $s \to \infty$ of this expression converges to 0, so $\mathcal{L}\{x\}$ has no pole at ∞. (This is something you must always check!) Thus the poles with largest real part are $\pm i$ and 0. Furthermore, the poles at $\pm i$ are simple (i.e. order 1), while the pole at 0 can be seen to be of order 4 (we have the s^3 in one denominator and the s in the other). Hence, we compute

$$\lim_{s \to \pm i} (s \mp i)\mathcal{L}\{x\} = \frac{-13 \mp 9i}{20}$$

$$\lim_{s \to 0} s^4 \mathcal{L}\{x\} = 2$$

We subtract off the $2/s^4$ term and continue to compute:

$$\lim_{s \to 0} s^3(\mathcal{L}\{x\} - 2/s^4) = 0$$

$$\lim_{s \to 0} s^2(\mathcal{L}\{x\} - 2/s^4) = -2$$

Hence $\mathcal{L}\{x\} - 2/s^4$ still had a pole at 0. This is the purpose of this check. But we must subtract that pole off and check again:

$$\lim_{s\to 0} s(\mathcal{L}\{x\} - 2/s^4 + 2/s^2) = 0$$

So finally we have found all the contributions of the poles of $\mathcal{L}\{x\}$ with 0 real part. We have that

$$\mathcal{L}\{x\} = \frac{2}{s^4} - \frac{2}{s^2} + \frac{-13-9i}{20(s-i)} + \frac{-13+9i}{20(s+i)} + F_0(s)$$

where F_0 has no poles with real part ≥ 0. Inverse Laplacing the terms besides F_0, we obtain the expression

$$\frac{1}{3}t^3 - 2t - \frac{13}{10}\cos t + \frac{9}{10}\sin t$$

By the Final Value Theorem, $x(t)$ grows at this rate as $t \to \infty$. Since $\frac{-13}{10}\cos t + \frac{9}{10}\sin t$ is bounded though, there's no need to include it (this term becomes miniscule as $t \to \infty$). Thus we conclude that

$$x(t) \sim \frac{1}{3}t^3 - 2t$$

as $t \to \infty$. □

Problem #7. Suppose that f, g and h are piecewise continuous functions with $|f(t)|, |g(t)|, |h(t)| \leq Ce^{\alpha t}$ for some $\alpha \in \mathbb{R}$. Define the *convolution* of f and g to be the function

$$(f * g)(t) = \int_0^t f(\tau)g(t-\tau)d\tau$$

(a) Show that $f * g = g * f$
(b) Show that $(f * g) * h = f * (g * h)$
(c) Show that $\mathcal{L}\{f * g\} = \mathcal{L}\{f\} \cdot \mathcal{L}\{g\}$.
(d) Use this to compute

$$\mathcal{L}^{-1}\left\{\frac{1}{(s^2+a^2)^2}\right\}.$$

Solution.
(a) This is the part that most people did correctly. It's just a change of variables.

$$(f * g)(t) = \int_0^t f(\tau)g(t-\tau)d\tau = -\int_t^0 f(t-\sigma)g(\sigma)d\sigma = \int_0^t g(\sigma)f(t-\sigma)d\sigma = (g * f)(t)$$

where I used $\sigma = t - \tau$.

(b) This is where people started having trouble. The key is to remember a basic fact from integration theory: Fubini's theorem! It says that under nice enough conditions (which are certainly assumed here) we can switch the order of integration, as long as the double integral is taken over the same domain. Here's what it comes down to in our case.

$$((f * g) * h)(t) = \left(\left(\int_0^t f(\tau)g(t-\tau)d\tau\right) * h\right)(t) = \int_0^t \left(\int_0^\sigma f(\tau)g(\sigma-\tau)d\tau\right) h(t-\sigma)d\sigma$$

$$= \int_0^t \left(\int_0^\sigma f(\tau) g(\sigma - \tau) h(t - \sigma) d\tau \right) d\sigma = \int_0^t \left(\int_\tau^t f(\tau) g(\sigma - \tau) h(t - \sigma) d\sigma \right) d\tau$$

The last equality sign is where I've used Fubeini's theorem. Notice that the bounds on each of these double integrals represent the same region of $\sigma\tau$-space. Continuing, we have

$$= \int_0^t f(\tau) \left(\int_\tau^t g(\sigma - \tau) h(t - \sigma) d\sigma \right) d\tau = \int_0^t f(\tau) \left(\int_0^{t-\tau} g(\omega) h(t - \tau - \omega) d\omega \right) d\tau$$

$$= \left(f * \left(\int_0^t g(\omega) h(t - \omega) d\omega \right) \right) (t) = (f * (g * h))(t)$$

This proves the claim. Note that I made the substitution $\omega = \sigma - \tau$ above.

(c) Again, people had trouble here, but again, this just comes down to using Fubini's theorem. Let's see how it works:

$$\mathcal{L}\{f * g\} = \int_0^\infty e^{-st} \left(\int_0^t f(\tau) g(t - \tau) d\tau \right) dt = \int_0^\infty \left(\int_\tau^\infty e^{-st} f(\tau) g(t - \tau) dt \right) d\tau$$

In the last equality, I used Fubini's theorem. Again, you should check that the two integrals are integrating over the same region of τt-space. Continuing,

$$= \int_0^\infty f(\tau) \left(\int_\tau^\infty e^{-st} g(t - \tau) dt \right) d\tau = \int_0^\infty f(\tau) \left(\int_0^\infty e^{-s(u+\tau)} g(u) du \right) d\tau$$

$$= \int_0^\infty e^{-s\tau} f(\tau) \left(\int_0^\infty e^{-su} g(u) du \right) d\tau = \int_0^\infty e^{-s\tau} f(\tau) \mathcal{L}\{g\} d\tau = \mathcal{L}\{g\} \int_0^\infty e^{-s\tau} f(\tau) d\tau$$

$$= \mathcal{L}\{g\}\mathcal{L}\{f\} = \mathcal{L}\{f\}\mathcal{L}\{g\}$$

This proves the claim. Note that I used the change of variables $u = t - \tau$.

(d) Using part (c), we know that

$$\mathcal{L}^{-1} \left\{ \frac{1}{(s^2 + a^2)^2} \right\} = \mathcal{L}^{-1} \left\{ \frac{1}{s^2 + a^2} \right\} * \mathcal{L}^{-1} \left\{ \frac{1}{s^2 + a^2} \right\} = \frac{1}{a^2} \int_0^t \sin(a\tau) \sin(a(t - \tau)) d\tau$$

After a long and tiring use of the entire table of trigonometric identities, one calculates that this integral is none other than

$$\frac{\sin at}{2a^3} - \frac{t \cos at}{2a^2}.$$

□