Homework 8 Sample Solutions

Problem #6. Let $\boldsymbol{F}: \mathbb{R}^n \to \mathbb{R}^n$ be C^1. Suppose that the autonomous system $\boldsymbol{X}' = \boldsymbol{F}(\boldsymbol{X})$ admits a global solution $\boldsymbol{X}(t)$ with $\boldsymbol{X}(t) = C\cos t\boldsymbol{E_1} + C\sin t\boldsymbol{E_2}$ for some $C > 0$, i.e. $\boldsymbol{X}(t)$ parameterizes a circle.

(a) Show that if $n = 2$ and $|\boldsymbol{X_0}| < C$, then the solution to the IVP

$$\begin{cases} \boldsymbol{X}' = \boldsymbol{F}(\boldsymbol{X}) \\ \boldsymbol{X}(0) = \boldsymbol{X}_0 \end{cases}$$

must satisfy $|\boldsymbol{X}(t)| < C$.

Solution. Most of you had the right idea on this one, but you must be rigorous! My solution is as follows:

Let $\boldsymbol{X}(t)$ be the solution to the IVP above. Suppose that $|\boldsymbol{X}(t_1)| \geq C$ for some t_1 (I'll assume $t_1 > 0$ for simplicity; the $t_1 < 0$ case is almost identical). Then $|\boldsymbol{X}(t)|$ is a continuous function, so there exists at $t_2 \in (0, t_1)$ such that $|\boldsymbol{X}(t_2)| = C$ by the intermediate value theorem.

Now, let $S = \{t \in (0, t_2) : |\boldsymbol{X}(t)| = C\}$, and let $t_3 = \inf S$ (the infimum used here is the greatest lower bound of this set). I claim that t_3 is actually in S. That is, I claim that $|\boldsymbol{X}(t_3)| = C$. To see this, note that for any $\varepsilon > 0$, $[t_3, t_3 + \varepsilon) \cap S \neq \emptyset$ (this is a property of the infimum). Thus, there always exists some $t \in [t_3, t_3 + \varepsilon) \cap S$, and by construction, $|\boldsymbol{X}(t)| = C$. Since t is arbitrarily close to t_3 and $|\boldsymbol{X}(t)|$ is continuous, it follows that $|\boldsymbol{X}(t_3)| = C$ as claimed.

Now consider the IVP given by

$$\begin{cases} \boldsymbol{Y'} = \boldsymbol{F}(\boldsymbol{Y}) \\ \boldsymbol{Y}(t_3) = \boldsymbol{X}(t_3) \end{cases}$$

(using our particular solution $\boldsymbol{X}$ that we have been considering).

Now, the local uniqueness theorem says that there exists *some* $a > 0$ for which this IVP has a unique solution on the domain $(t_3 - a, t_3 + a)$. On the one hand, our $\boldsymbol{X}(t)$ is clearly such a solution. On the other hand, we have the solution $\boldsymbol{X}_{\text{circ}}(t) = C\cos t\boldsymbol{E_1} + C\sin t\boldsymbol{E_2}$. This is not necessarily a solution of the above IVP, but we can change variables to make it so. After all, there exists some t_4 for which $\boldsymbol{X}_{\text{circ}}(t_4) = \boldsymbol{X}(t_3)$. Then we simply consider $\boldsymbol{Y}(t) = \boldsymbol{X}_{\text{circ}}(t + t_4 - t_3)$. By construction, $\boldsymbol{Y}$ is another solution to the IVP, and $|\boldsymbol{Y}(t)| = C$ for all t. But $|\boldsymbol{X}(t_3 - a/2)| < C$ since t_3 is the infimum of S.

Thus, $\boldsymbol{X}$ and $\boldsymbol{Y}$ differ at the point $t_3 - a/2$, which is in $(t_3 - a, t_3 + a)$, which contradicts the local uniqueness theorem. Hence, we conclude that $|\boldsymbol{X}(t)| < C$ for all t. $\square$

Problem #7. Show that if $u : [a, b] \to \mathbb{R}$ is a C^1 function that satisfies the differential *inequality*

$$u' \leq \mu u + g(t),$$

where g is continuous, then, for $t \in [a, b]$,

$$u(t) \leq u(a)e^{\mu(t-a)} + \int_a^t e^{\mu(t-s)} g(s)ds.$$

Solution. This follows the same method as solving a differential equation with an integrating factor. First, we move a term over.

$$u' - \mu u \leq g(t)$$

Next, we multiply both sides by $e^{-\mu t}$. This does not change the inequality sign because $e^{-\mu t}$ is always positive. We get

$$u' e^{-\mu t} - \mu u e^{-\mu t} = \frac{d}{dt}\left(u e^{-\mu t}\right) \leq g(t)e^{-\mu t}$$

The equality on the left is just the product rule. Now, there is a property of integrals that if $c \leq d$ and $f(t) \leq h(t)$ for all $t \in (a, b)$, then $\int_c^d f(t)dt \leq \int_c^d h(t)dt$. Applying this to the above, we get

$$\int_a^t \frac{d}{ds}\left(u(s)e^{-\mu s}\right) ds = u(t)e^{-\mu t} - u(a)e^{-\mu a} \leq \int_a^t g(s)e^{-\mu s} ds$$

for any $t \in [a, b]$. Adding a term back and multiplying by $e^{\mu t}$, which is positive for all t, we get

$$u(t) \leq u(a)e^{\mu(t-a)} + \int_a^t e^{\mu(t-s)} g(s)ds$$

as desired. □

Problem #8. Show that if $u : [a, b) \to \mathbb{R}$ is a *positive* C^1 function that satisfies the differential inequality

$$u' \geq \mu u^2$$

for $\mu > 0$, then we must have $b \leq a + \frac{1}{u(a)\mu}$.

Solution. Since $u^2 > 0$, we can divide by it without changing inequality signs. We get

$$\frac{u'}{u^2} \geq \mu$$

Integrate both sides from a to $t \in [a, b)$ to get

$$\frac{1}{u(a)} - \frac{1}{u(t)} \geq \mu(t - a)$$

Note that since $\frac{1}{u(t)} > 0$, we have

$$\frac{1}{u(a)} \geq \frac{1}{u(a)} - \frac{1}{u(t)} \geq \mu(t-a)$$

for all $t \in [a, b)$. Taking the limit as $t \to b$, we get $\frac{1}{u(a)} \geq \mu(b-a)$. Multiply each side by the positive quantity $\frac{u(a)}{b-a}$ to obtain

$$\frac{1}{b-a} \geq u(a)\mu$$

Since both sides of the inequality are positive, inversion flips the inequality sign, hence

$$b - a \leq \frac{1}{u(a)\mu}$$

The desired result follows. □

Problem #9. Use Problem #7, the Cauchy-Schwarz inequality, and the theorem on pg. 146 to show that if

$$\boldsymbol{F} : \mathbb{R}^n \to \mathbb{R}^n$$

is C^1 and satisfies $|\boldsymbol{F}(\boldsymbol{X})| \leq C|\boldsymbol{X}|$ for some $C > 0$, then the IVP

$$\begin{cases} \boldsymbol{X'} = \boldsymbol{F}(\boldsymbol{X}) \\ \boldsymbol{X}(0) = \boldsymbol{X}_0 \end{cases}$$

has a global solution.

Solution. Let $u(t) = |\boldsymbol{X}(t)|^2$. The first thing to notice is that

$$u'(t) = \frac{d}{dt}|\boldsymbol{X}(t)|^2 = \frac{d}{dt}\left(\boldsymbol{X}(t) \cdot \boldsymbol{X}(t)\right) = 2\boldsymbol{X'}(t) \cdot \boldsymbol{X}(t)$$
$$= 2\boldsymbol{F}(\boldsymbol{X}) \cdot \boldsymbol{X} \leq 2|\boldsymbol{F}(\boldsymbol{X}) \cdot \boldsymbol{X}| \leq 2|\boldsymbol{F}(\boldsymbol{X})|\,|\boldsymbol{X}|$$

where we've used Cauchy-Schwarz at the last step. Using what we know about $\boldsymbol{F}$, this says that $u' \leq 2C|\boldsymbol{X}|\,|\boldsymbol{X}| = 2Cu$.

Now, we know that there exists a maximal interval on which the solution $\boldsymbol{X}(t)$ exists. Call it (α, β), and suppose that $\beta < \infty$. Consider any closed interval $[a, b] \subset (\alpha, \beta)$. By Problem #7, the above implies that

$$u(t) \leq u(a)e^{2C(t-a)}$$

As a consequence, $|\boldsymbol{X}(t)| \leq u(a)e^{C(t-a)}$ for all $t \in [a, b]$. Since b could be anything in (a, β), this indeed holds for all $t \in [a, \beta)$. This implies that $\lim_{t \to \beta} u(t) < \infty$, which contradicts the theorem on pg. 146. Hence, we conclude that $\beta = \infty$. A similar argument shows that $\alpha = -\infty$, hence the claim is proved. □