Homework 9 Sample Solutions

Problem 8.5. Consider the system

$$\begin{cases} x' = x^2 + y \\ y' = x - y + a \end{cases}$$

where a is a parameter.

(a) Find all equilibrium points and compute the linearized equation at each.

(b) Describe the behavior of the linearized system at eacu equilibrium point.

(c) Describe any bifurcations that occur.

Solution.

(a) We set $x' = y' = 0$ and find the solutions. From $y' = 0$, we find that $y = x + a$. Plugging this in to the $x' = 0$ equation, we find $x^2 + x + a = 0$. The quadratic formula yields $x = \frac{-1 \pm \sqrt{1-4a}}{2}$. Since $y = x + a$, we get $y = \frac{-1+2a \pm \sqrt{1-4a}}{2}$ (where the $\pm$ signs for x and y are chosen to be the same). Thus, we have two equilibrium points if $a < 1/4$, one if $a = 1/4$, and none if $a > 1/4$.

Taking the Jacobian of the appropriate function $F(x, y)$, we get $DF_{(x,y)} = \begin{pmatrix} 2x & 1 \\ 1 & -1 \end{pmatrix}$. Plugging in our equilibria, we find that $DF = \begin{pmatrix} -1 \pm \sqrt{1-4a} & 1 \\ 1 & -1 \end{pmatrix}$. It's obvious how to get the linearized systems at the equilibria from this information.

(b) For the point $(\frac{-1+\sqrt{1-4a}}{2}, \frac{-1+2a+\sqrt{1-4a}}{2})$, the determinant of the matrix of the linearized system is $-\sqrt{1-4a}$. This is negative if $a < 1/4$, hence by Figure 4.1 of the text (page 64), this linear system has a saddle point.

For the point $(\frac{-1-\sqrt{1-4a}}{2}, \frac{-1+2a-\sqrt{1-4a}}{2})$, the trace and determinant of the matrix of the linearized system are $T = -2 - \sqrt{1-4a}$ and $D = \sqrt{1-4a}$ respectively. Thus the determinant is positive for $a < 1/4$, but $T^2 - 4D = 5 - 4a > 0$ for $a < 1/4$. Thus, by Figure 4.1, the linearized system has a nodal sink.

When $a = 1/4$, the linearized systems are the same (as there's only one equilibrium point), and once can explicitly find that the linearized system has eigenvalues $0, -2$.

(c) It is clear that there is a bifurcation at the point $a = 1/4$, since the number of equilibria changes as a changes through this value.

□

Problem 9.2. Describe the phase portrait for

$$\begin{cases} x' = x^2 - 1 \\ y' = -xy + a(x^2 - 1) \end{cases}$$

when $a < 0$. What qualitative features of this flwo changes as a passes from positive to negative?

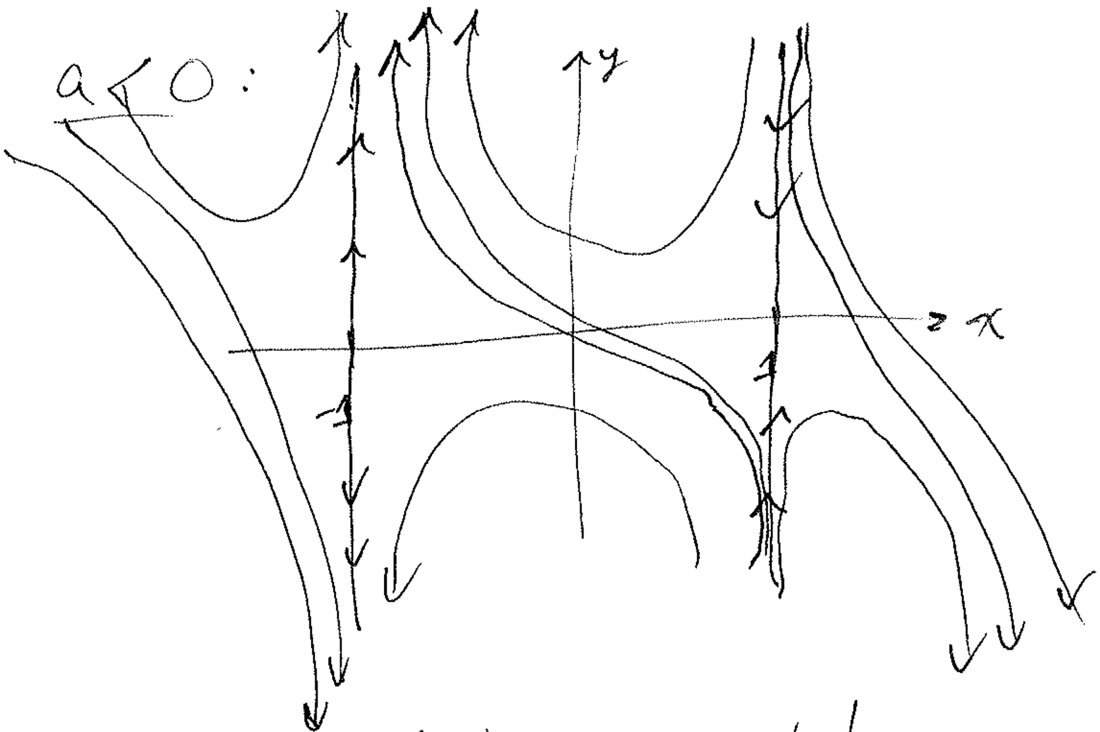
Solution. It is clear that we have x-nullclines on the vertical lines $x = -1$ and $x = 1$. By plugging in points of the form $(-1, y)$ and $(1, y)$ into our system (the value of a doesn't matter at these points), we find that these vertical lines are a stable line for $(1, 0)$ and an unstable line for $(-1, 0)$.

Now consider points $(x, 0)$, where $-1 < x < 1$. Plugging these in to our system, we find that $x^2 - 1 < 0$ in this range, so at these points $x' < 0$. Further, $y' > 0$ if $a < 0$, $y' = 0$ if $a = 0$, and $y' < 0$ if $a > 0$.

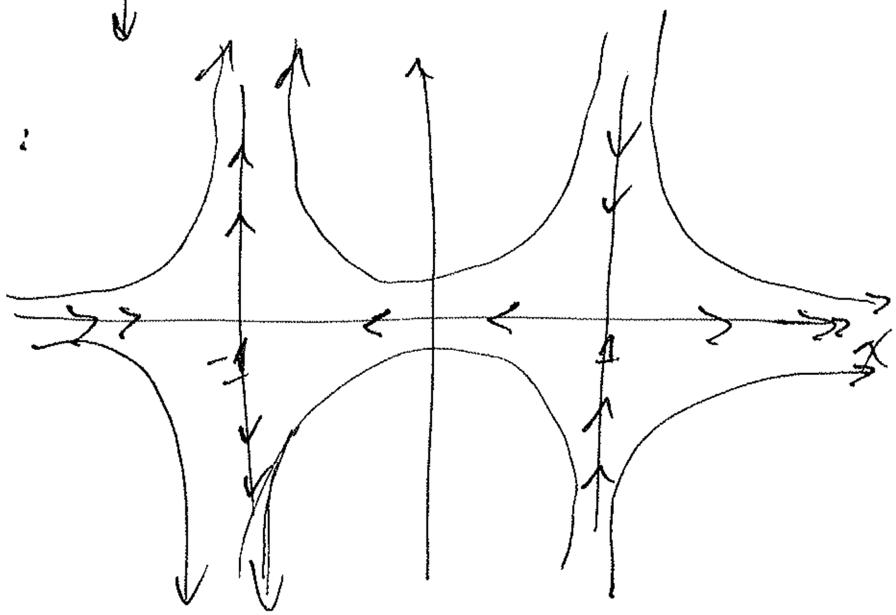
Finally, consider the rest of the x-axis. For any a, we find that $x' > 0$ here. Moreover, y' and a have the same sign for all a. Using this information, we obtain the sketches attached to this document.

From the sketches, we can see that for $a < 0$, we have solutions on the far left and far right which extend from $y = \infty$ to $y = -\infty$. In the middle strip, we have solutions extending from $y = -\infty$ to $y = \infty$. For $a = 0$, the x-axis becomes a barrier (it is part of the y-nullcline) and all solutions are restricted to either the upper or lower half plane. Finally, for $a > 0$, we have solutions on the far left and far right extending from $y = -\infty$ to $y = \infty$, and in the middle strip, solutions extend from $y = \infty$ to $y = -\infty$. Thus, the solutions have apparently flipped with respect to the x-axis. □

a < 0 :



a = 0 :



a > 0 :