Math 405 Final Exam — December 11, 2019

Name: ____

- Complete the following problems. In order to receive full credit, please make sure to *justify your answers*. You are free to use results from class or the course textbook as long as you clearly state what you are citing.
- **You have 3 hours.** This is a closed-book, closed-notes exam. No calculators or other electronic aids will be permitted (nor are they needed). If you finish early, you must hand your exam paper to a proctor.
- Please check that your copy of this exam contains 12 numbered pages and is correctly stapled.
- If you need extra room, use the back sides of each page. If you must use extra paper, make sure to write your name on it and attach it to this exam. Do not unstaple or detach pages from this exam.

The following boxes are strictly for grading purposes. Please do not mark.

Question	Points	Score
1	30	
2	30	
3	35	
4	35	
5	35	
6	35	
Total	200	

1. (a) (10 points) State the Min/Max Theorem (also called the Extreme Value Theorem).

   (b) (5 points) Give an example of a continuous function $f : (-1, 1) \to \mathbb{R}$ that achieves its maximum value, but does not achieve its minimum value.

(c) (15 points) Show that if $f : [0, 1] \to (0, 1)$ is continuous, then f is *not* onto.

2. (a) (10 points) State the formal definition of uniform continuity of a function $f : (a, b) \to \mathbb{R}$.

   (b) (5 points) Give an example of a continuous function $f : (a, b) \to \mathbb{R}$ that is not bounded. You do not need to justify your answer.

(c) (15 points) Show that if $f : (a,b) \to \mathbb{R}$ is uniformly continuous, then there is a number M so $|f(x)| \leq M$ for all $x \in (a,b)$. That is, f is bounded.

3. (a) (5 points) State the definition of a function $f : (a, b) \to \mathbb{R}$ being strictly increasing on (a, b).

(b) (10 points) Show that if $f : (a, b) \to \mathbb{R}$ is differentiable and $f'(x) > 0$ for all $x \in (a, b)$, then f is strictly increasing on (a, b).

(c) (10 points) Show that if $f : (a,b) \to \mathbb{R}$ is strictly increasing and $c \in (a,b)$, then $\lim_{x\to c_-} f(x)$ and $\lim_{x\to c_+} f(x)$ both exist and satisfy $\lim_{x\to c_-} f(x) \le f(c) \le \lim_{x\to c_+} f(x)$.

(d) (10 points) Show that if $g : (a,b) \to \mathbb{R}$ is differentiable and $g' : (a,b) \to \mathbb{R}$ is strictly increasing, then g' is continuous. (Hint: Recall, the derivative of a differentiable function has the intermediate value property).

4. (a) (10 points) Show that if $f : [a, b] \to [0, 1]$ satisfies $f(x) = 0$ for all $x \in [a, b] \cap \mathbb{Q}$, then

$$\underline{\int_a^b} f(x)dx = 0.$$

That is the lower Darboux integral of f vanishes.

(b) (5 points) Give an example of a discontinuous function $f : [0, 1] \to \mathbb{R}$ that is Riemann integrable.

(c) (20 points) Let $f : (a, b) \to \mathbb{R}$ be uniformly continuous. Show directly from definitions that if $g : [a, b] \to \mathbb{R}$ satisfies $g(x) = f(x)$ for $x \in (a, b)$, then g is Riemann integrable.

5. (a) (15 points) State both versions of the Fundamental Theorem of Calculus.

(b) (5 points) Give an example of a function $f : [-1, 1] \to \mathbb{R}$ that is Riemann integrable, but $F(x) = \int_0^x f(t)dt$ is not differentiable at $x = 0$.

(c) (15 points) Show that if $f : (-1, 1) \to \mathbb{R}$ is C^1 with $f(0) = 0$ and $f'(x) \geq 2|x|$, then $|f(x)| \geq x^2$ for all $x \in (-1, 1)$.

6. (a) (10 points) State the definition of a sequence of functions $f_n : [a, b] \to \mathbb{R}$ uniformly converging to $f : [a, b] \to \mathbb{R}$.

(b) (5 points) Give an example of a sequence of functions $f_n : [0, 1] \to \mathbb{R}$ so that f_n converges pointwise to $f : [0, 1] \to \mathbb{R}$ but not uniformly.

(c) (20 points) Prove that if $f_n : [a, b] \to \mathbb{R}$ are continuous and the f_n converge uniformly to $f : [a, b] \to \mathbb{R}$, then f is continuous.